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RESEARCH ARTICLE

A MODEL OF COMPETITION BETWEEN PERPETUAL SOFTWARE AND SOFTWARE AS A SERVICE

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Appendix A

Modeling Notations

Table A1. Modeling Notations

Notation	Definition
$t \in [0, 1]$	Time within the software life cycle $[0,1]$
q	Quality of the old perpetual software product
ρ	New perpetual software quality improvement ratio over the old version
θ	The SaaS initial quality improvement ratio over the old perpetual software, $1 < \theta < \rho$
α	Rate of software quality improvement for the SaaS product
p_u	One-time upgrade price for existing users to upgrade to the new perpetual software
p_n	One-time purchase price for new users to buy the new perpetual software
p_s	The SaaS price for per unit time use of the software
n_t	The network size at time t, where $n_t = \{1, 2\}$
k	Marginal network effect
δ	Perpetual software incremental quality improvement ratio over the old version
c_α	The SaaS vendor's quality improvement cost per unit time
c	OG users' cost of switching to SaaS

Appendix B

Elimination of Strategy Pairs in Table 1

Given the software quality improvement $\rho q > q$, the OG consumers are willing to pay a positive price to upgrade to the new perpetual software. Because all software development costs have been sunk, the perpetual software vendor can always sell to the OG users at a positive price to earn non-zero profit. So in equilibrium, any strategy pair that involves the OG users that continue to use the old version of perpetual software is dominated by other induced user strategies. We therefore eliminate the first row of strategy pairs in Table 1.

Similarly, (Old + SaaS, SaaS) and (SaaS, SaaS) can be eliminated because the perpetual software vendor earns zero profit. Because the perpetual software has the quality advantage over the SaaS at time 0, the perpetual software vendor, by charging a very small positive upgrade price ε, is able to induce the OG consumers to upgrade and earn a non-zero profit.

Also note that if the OG users choose SaaS, the NG users prefer SaaS as well. The reason is that the OG users are more "sticky" to the perpetual software than the NG users because of their reserve utility from the old perpetual software. Therefore, neither (SaaS, New) nor (SaaS, New + SaaS) can achieve and sustain equilibrium.

Finally, once both OG and NG users adopt the new version perpetual software, they become identical. They should take the same action afterward—either they both continue to use the new version or they switch to SaaS at some time point simultaneously. This rules out (Upgrade, New + SaaS) and (Upgrade + SaaS, New). As a result, only six strategy pairs, SP1 ~ SP6, are possible in equilibrium.

Appendix C

Parameter Configuration for Strategy Pairs SP1 ~ SP6

Figure C1 graphically shows how the six possible strategy pairs can be supported by different combinations of the SaaS quality improvement rate and the SaaS price. The parameter configurations for each strategy pair are presented in Table C1. We observe that the network effect will affect the appearance of SP2, SP4, and SP5. When the network effect is stronger, users tend to choose the same type of software; that is, when the dashed line in Figure C1 shifts up to the left, the appearance of SP2 becomes less likely, while that of E4 and E5 becomes more likely.

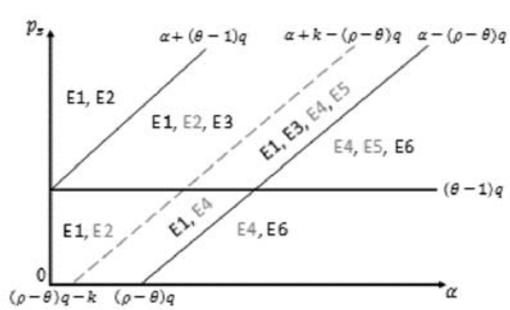

Figure C1. Possible Outcomes and Feasible Regions

Table C1. Parameter Configuration for Each Strategy Pair

Strategy Pair	Feasible Conditions
SP1 (Upgrade, New)	$p_s \geq \alpha - (\rho - \theta)q$
SP2 (Upgrade, SaaS)	$p_s \geq \alpha + k - (\rho - \theta)q$
SP3 (Old+SaaS, New)	$\max[(\theta - 1)q, \alpha + k - (\rho - \theta)q] \leq p_s \leq \alpha + (\theta - 1)q$
SP4 (Upgrade+SaaS, SaaS)	$p_s \leq \alpha + k - (\rho - \theta)q$
SP5 (Old+SaaS, New+SaaS)	$(\theta - 1)q \leq p_s \leq \alpha + k - (\rho - \theta)q$
SP6 (Upgrade+SaaS, New+SaaS)	$p_s \leq \alpha - (\rho - \theta)q$

SP1: Because both groups adopt the new perpetual software, they are identical after adoption. In SP1, no groups switch to SaaS over the entire software life cycle, implying that the SaaS payoff at the end of the software life cycle is no higher than the new perpetual software. Hence, $\theta q + \alpha + 2k - p_s \leq \rho q + 2k$, which leads to $p_s \geq \alpha - (\rho - \theta)q$.

SP2: To prevent the OG users from switching to SaaS, the SaaS payoff at the end of the software life cycle should not be higher than payoff from the new perpetual software for OG users. Note that, without switching, the OG users derive the network utility k; if switching, they can enjoy the network utility $2k$ because the NG users have adopted SaaS. Hence, $\theta q + \alpha + 2k - p_s \leq \rho q + k$, which leads to $p_s \geq \alpha + k - (\rho - \theta)q$.

SP3: For the OG users to switch but for NG users not to switch during the software life cycle, we have three conditions: (1) the OG users prefer the old perpetual software rather than SaaS at time 0 (i.e., $\theta q + k - p_s \leq q + k$); (2) the OG users prefer SaaS rather than the old perpetual software at the end of the software life cycle (i.e., $\theta q + \alpha + k - p_s \geq q + k$); and (3) the NG users prefer the new perpetual software rather than SaaS at the end of the software life cycle (i.e., $\theta q + \alpha + 2k - p_s \leq \rho q + k$). All together, we have $\max[(\theta - 1)q, \alpha + k - (\rho - \theta)q] \leq p_s \leq \alpha + (\theta - 1)q$.

SP4: For switching to occur, OG users derive higher payoff from SaaS than from the new perpetual software at the end of the software life cycle. Hence, $\theta q + \alpha + 2k - p_s \geq \rho q + k$, which leads to $p_s \leq \alpha + k - (\rho - \theta)q$.

SP5: We have two conditions: (1) the OG users prefer the old perpetual software rather than SaaS at time 0 (i.e., $\theta q + k - p_s \leq q + k$); and (2) the NG users derive higher payoff from SaaS than from the new perpetual software at the end of the software life cycle (i.e., $\theta q + \alpha + 2k - p_s \geq \rho q + k$). Therefore, $(\theta - 1)q \leq p_s \leq \alpha + k - (\rho - \theta)q$.

SP6: Note that both OG and NG users must switch at the same time. They derive higher payoff from SaaS than from the new perpetual software at the end of the software life cycle. Hence, $\theta q + \alpha + 2k - p_s \geq \rho q + 2k$, which leads to $p_s \leq \alpha - (\rho - \theta)q$.

Appendix D

Baseline Model Equilibrium Outcomes

Table D1 presents vendors' optimal prices, profit, consumer surplus, and social welfare under each equilibrium in the baseline model.

Table D1. Equilibrium Prices, Profits, Consumer Surplus, and Social Welfare: Baseline Model

(a) Equilibrium Prices: Baseline Model

Equilibrium	p_u	p_n	p_s
Monopoly (M)	$(\rho-1)q+k$	$\rho q+2k$	NA
Entry Deterrence (I)	$(\rho-\theta)q-\frac{\alpha}{2}+k$	$(\rho-\theta)q-\frac{\alpha}{2}+k$	0
Market Segmentation (IIa)	$(\rho-1)q$	$(\rho-1)q+2k$	$(\theta-1)q+k+\frac{\alpha}{2}$
Market Segmentation IIb)	$(\rho-1)q$	$\frac{\alpha}{2}+2k$	$\alpha+k-(\rho-\theta)q$
Sequential Dominance (IIIa)	$\frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$	$\frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$	$\frac{\alpha-(\rho-\theta)q}{2}$
Sequential Dominance (IIIb)	$\frac{\alpha k+[\alpha(\rho-1)+k(\rho-\theta)]q-(\rho-1)(\theta-1)q^2}{6\alpha}$	$\frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$	$\frac{\alpha-(\rho-\theta)q}{2}$

(b) Equilibrium Profits: Baseline Model

Equilibrium	π_{perp}	π_{SaaS}
Monopoly (M)	$2(\rho-1)q+3k$	NA
Entry Deterrence (I)	$2(\rho-\theta)q-\alpha+2k$	0
Market Segmentation (IIa)	$(\rho-1)q$	$(\theta-1)q+k+\frac{\alpha}{2}$
Market Segmentation IIb)	$(\rho-1)q$	$\alpha+k-(\rho-\theta)q$
Sequential Dominance (IIIa)	$\frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{4\alpha}$	$\frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$
Sequential Dominance (IIIb)	$\frac{2[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]-[\alpha-(\rho+\theta-2)q]^2}{8\alpha}$	$\frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$

(c) Equilibrium Consumer Surplus and Social Welfare: Baseline Model

Equilibrium	CS_{OG}	CS_{NG}	SW
Monopoly (M)	$q+k$	0	$2\rho q+4k$
Entry Deterrence (I)	$\theta q+k+\frac{\alpha}{2}$	$\theta q+k+\frac{\alpha}{2}$	$2\rho q+4k$
Market Segmentation (IIa)	$q+k$	q	$(\rho+\theta)q+2k+\frac{\alpha}{2}$
Market Segmentation IIb)	$q+k$	$\rho q-\frac{\alpha}{2}$	$(\rho+\theta)q+2k+\frac{\alpha}{2}$
Sequential Dominance (IIIa)	$\frac{3\alpha k+[\alpha(\rho+\theta)-k(\rho-\theta)]q}{2\alpha}$	$\frac{3\alpha k+[\alpha(\rho+\theta)-k(\rho-\theta)]q}{2\alpha}$	$\frac{3\alpha^2+16\alpha k+2\alpha(\rho+3\theta)q+3(\rho-\theta)^2q^2}{4\alpha}$
Sequential Dominance (IIIb)	$\frac{\alpha^2+12\alpha k+[2\alpha(\rho+\theta+2)-4k(\rho-\theta)]q+(\rho+\theta-2)^2q^2}{8\alpha}$	$\frac{3\alpha k+[\alpha(\rho+\theta)-k(\rho-\theta)]q}{2\alpha}$	$\frac{3\alpha^2+16\alpha k+2\alpha(\rho+3\theta)q+3(\rho-\theta)^2q^2}{4\alpha}$

Appendix E

Proofs for Baseline Model

Proof of Proposition 1 (Monopoly Market Equilibrium)

Proof. When no entry threat arises from the SaaS vendor, the perpetual software vendor is the monopolist. When the vendor releases the new version software at time 0, it charges a purchase price to the NG users so that it extracts all surpluses from them, and so $p_n^M = \rho q + 2k$. Meanwhile, it charges an upgrade price p_u as high as possible to induce the OG users to upgrade to the new version (i.e., $\rho q + 2k - p_u \geq q + k$)). Therefore, $p_n^M = (\rho - 1)q + k$. The vendor's profit is $\pi^M = p_u^M + p_n^M = (2\rho - 1)q + 3k$.

Proof of Proposition 2 (Entry Deterrence Equilibrium)

Proof. This is the case in which $\alpha \leq (\rho - \theta)q$. Because the SaaS quality is always lower than the new perpetual software, users do not switch. The perpetual software vendor can choose either the entry deterrence strategy to serve both user groups and drive the SaaS vendor out of the market or it can choose the market segmentation strategy and serve OG users only. The equilibrium strategy pair corresponding to the former case is SP1 (Upgrade, New), while in the latter case it is SP2 (Upgrade, SaaS).

Consider SP1 (Upgrade, New). Given that NG users adopt the new version perpetual software, the OG users have three strategies to consider. If they keep using the old version, their total utility is $q + k$; if OG users choose the SaaS at time 0, their total utility is $\int_0^1 (\theta q + \alpha t + k)dt$; and if OG users choose to upgrade and then keep using the new perpetual software, their total utility is $\rho q + 2k - p_u$.

To ensure that the OG users prefer upgrading to the new version rather than continuing to use the old version, their total utility must be $\rho q + 2k - p_u \geq q + k$, which is $p_u \leq (\rho - 1)q + k$ (IC1). Meanwhile, the perpetual software vendor needs to make sure that OG users prefer upgrading rather than adopting SaaS, even if the SaaS price is reduced to zero. That is, the entry deterrence condition is $\rho q + 2k - p_u \geq \int_0^1 (\theta q + \alpha t + k)dt$, and it gives $p_u \leq (\rho - \theta)q + k - \frac{\alpha}{2}$ (IC2). We can show that (IC1) is not binding.

Similarly, given that OG users choose to upgrade, the NG users' total utility is $\rho q + 2k - p_n$ if they choose the new perpetual software and $\int_0^1 (\theta q + \alpha t + k)dt$ if they opt for SaaS at time 0 at zero price. To ensure that the NG users prefer the new perpetual software to the SaaS, even if the SaaS price is zero, their total utility must be $\rho q + 2k - p_n \geq \int_0^1 (\theta q + \alpha t + k)dt$; that is, $p_n \leq (\rho - \theta)q + k - \frac{\alpha}{2}$ (IC3).

Because $p_u \leq p_n$, by (IC2) and (IC3) the perpetual software vendor sets the prices at respective upper bounds: $p_n^{SP1} = p_u^{SP1} = (\rho - \theta)q + k - \frac{\alpha}{2}$. Consequently, we obtain the perpetual software vendor' s profit at $\pi_{perp}^{SP1} = 2(\rho - \theta)q + 2k - \alpha$, and the SaaS vendor is out of the market.

Finally, we need to prove that the perpetual software vendor earns a higher profit under SP1 than SP2, which is true when $k \geq K_1 = \frac{\alpha - (\rho - 2\theta + 1)q}{2}$, as shown in the proof of Proposition 3. Hence, the perpetual software vendor deters the SaaS vendor's entry when $k \geq K_1$.

Proof of Proposition 3 (Market Segmentation Equilibrium—α Low)

Proof. Consider SP2 (Upgrade, SaaS). Given that the NG users adopt SaaS, if the OG users continue to use the old version perpetual software, their total utility is $q + k$; if the OG users choose SaaS, the total utility is $\int_0^1 (\theta q + \alpha t + 2k - p_s) dt$; and if they choose to upgrade and then continue to use the new perpetual software over the entire software life cycle, the total utility is $\rho q + k - p_u$.

To ensure the OG users prefer to upgrade rather than to continue to use the old version, their total utility must be $\rho q + k - p_u \geq q + k$ and thus $p_u \leq (\rho - 1) q$ (IC4). Also, to ensure that the OG users prefer to upgrade rather than opt for SaaS, their total utility must be $\rho q + k - p_u \geq \int_0^1 (\theta q + \alpha t + 2k - p_s) dt$ and thus $p_s \geq p_u - (\rho - \theta) q - k + \frac{\alpha}{2}$ (IC5).

Similarly, given that OG users upgrade, the NG users' total utility is $\rho q + 2k - p_n$ if they choose the new perpetual software and $\int_0^1 (\theta q + \alpha t + k - p_s) dt$ if they opt for SaaS at time 0. To ensure that the NG users prefer SaaS, their total utility must be $\rho q + 2k - p_n \leq \int_0^1 (\theta q + \alpha t + k - p_s) dt$; that is, $p_s \leq p_n - (\rho - \theta) q - k + \frac{\alpha}{2}$ (IC6).

To maximize its profit, the perpetual software vendor sets p_n as high as possible so that the SaaS vendor can also charge a high enough price p_s, which in turn allows the perpetual software vendor to charge a high upgrade price p_u. As a result, the perpetual software vendor charges $p_u^{SP2} = (\rho - 1) q$ to make the OG users' IC constraint (IC4) binding. It sets $p_n^{SP2} = (\rho - 1) q + 2k$ so that the SaaS vendor charges the highest possible $p_s^{SP2} = (\theta - 1) q + k + \frac{\alpha}{2}$ by (IC6) that does not violate (IC5). Finally, under the condition $\alpha < (\rho - \theta) q$, we can verify that the condition for SP2, $p_s > \alpha + k - (\rho - \theta) q$ as specified in Table C1, holds.

Finally, we need to show that the perpetual software vendor's profit under SP2, $\pi_{perp}^{SP2} = (\rho - 1) q$, is higher than its profit under SP1. Solving $\pi_{perp}^{SP2} > \pi_{perp}^{SP1}$, we have $k < K_1$, where K_1 is defined in Proposition 2. Hence, SP2 (Upgrade, SaaS) sustains as an equilibrium user strategy pair when $k < K_1$. Also note that $K_1 = 0$ when $\alpha = (\rho - 2\theta + 1) q \doteq \underline{\alpha}$.

Proof of Proposition 4 (Sequential Dominance Equilibrium)

Proof. Consider SP6 (Upgrade+SaaS, New+SaaS). The switching time t_{s3} is determined by $\theta q + \alpha t_{s3} + 2k - p_s = \rho q + 2k$, so that $t_{s3} = \frac{p_s + (\rho - \theta) q}{\alpha}$. The SaaS vendor's profit is expressed as $2 p_s \left(1 - \frac{p_s + (\rho - \theta) q}{\alpha}\right)$. Solving this optimization problem yields the optimal SaaS price $p_s^* = \frac{\alpha - (\rho - \theta) q}{2}$. We can verify that p_s^* satisfies the SP6 condition in Table 3. Consequently, $t_{s3}^* = \frac{\alpha + (\rho - \theta) q}{2\alpha}$. Several incentive compatibility conditions must be satisfied, as follows.

Given that the OG users choose Upgrade+SaaS, the NG users prefer New+SaaS rather than SaaS if $(\rho q + 2k) t_{s3}^* - p_n + \int_{t_{s3}^*}^1 \left(\theta q + \alpha t + 2k - p_s^*\right) dt \geq \int_0^{t_{s3}^*} \left(\theta q + \alpha t + k - p_s^*\right) dt + \int_{t_{s3}^*}^1 \left(\theta q + \alpha t + 2k - p_s^*\right) dt$. So $p_n \leq \frac{[\alpha + (\rho - \theta) q][4k + \alpha + (\rho - \theta) q]}{8\alpha}$ (IC7).

Given that the NG users choose New+SaaS, the OG users prefer Upgrade+SaaS rather than Old+SaaS if $(\rho q + 2k)t_{s2}^* - p_u + \int_{t_{s2}^*}^{1} (\theta q + \alpha t + 2k - p_s^*)dt \geq (q + k)t_{s1}^* + \int_{t_{s1}^*}^{t_{s2}^*} (\theta q + \alpha t + k - p_s^*)dt + \int_{t_{s2}^*}^{1} (\theta q + \alpha t + 2k - p_s^*)dt$. The condition gives $p_u \leq \frac{k\alpha-(\rho-1)(\theta-1)q^2+[\alpha(\rho-1)+k(\rho-\theta)]q}{2\alpha}$ (IC8). Note that the switching time $t_{s2}^* = t_{s3}^*$. The switching time t_{s1}, for Old+SaaS, is determined by $\theta q + \alpha t_{s1} + k - p_s = q + k$, so that $t_{s1} = \frac{p_s-(\theta-1)q}{\alpha}$. Substituting p_s^* into the expression of t_{s1}, we have $t_{s1}^* = \frac{\alpha-(\rho+\theta-2)q}{2\alpha}$.

If $\alpha \leq (\rho + \theta - 2)q$, $t_{s1}^* < 0$, so that OG users prefer SaaS. To ensure the OG users prefer Upgrade+SaaS rather than SaaS, we need $(\rho q + 2k)t_{s3}^* - p_u + \int_{t_{s3}^*}^{1} (\theta q + \alpha t + 2k - p_s^*)dt \geq \int_{0}^{t_{s3}^*} (\theta q + \alpha t + k - p_s^*)dt + \int_{t_{s3}^*}^{1} (\theta q + \alpha t + 2k - p_s^*)dt$; that is, $p_u \leq \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$ (IC9). So by (IC7) and (IC9) we have $p_u^{SP6} = p_n^{SP6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$, and the perpetual software vendor's profit is $\pi_{perp}^{SP6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{4\alpha}$.

If $\alpha > (\rho + \theta - 2)q$, $t_{s1}^* > 0$, by (IC7) and (IC8) we have $p_u^{SP6} = \frac{k\alpha-(\rho-1)(\theta-1)q^2+[\alpha(\rho-1)+k(\rho-\theta)]q}{2\alpha} < p_n^{SP6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$, and $\pi_{perp}^{SP6} = \frac{2[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]-[\alpha-(\rho+\theta-2]^2}{8\alpha}$.

Under both cases, the SaaS price is $p_s^{SP6} = \frac{\alpha-(\rho-\theta)q}{2}$, and the SaaS vendor's profit is $\pi_{SaaS}^{SP6} = \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$.

Another outcome under the strategy pair SP2 (Upgrade, SaaS) is solved in Proposition 5. Comparing the two vendors' respective profits under SP2 and SP6, we show that when the network effect k is stronger than a threshold value K_2 (details in the proof of Proposition 5), SP6 (Upgrade+SaaS, New+SaaS) emerges as the final equilibrium user strategy.

Proof of Proposition 5 (Market Segmentation Equilibrium—α High)

Proof. Consider SP2 (Upgrade, SaaS). The analysis is similar to the proof for Proposition 3. The only difference is that when $\alpha > 2(\rho - 1)q$, the constraint $p_s \geq \alpha + k - (\rho - \theta)q$ (refer to Table 3) is binding. Therefore, $p_s^{SP2} = \alpha + k - (\rho - \theta)q$ if $\alpha > 2(\rho - 1)q$. Also, we need to reexamine the IC conditions. (IC5) becomes $p_u \leq \frac{\alpha}{2}$. Because $(\rho - 1)q \leq \frac{\alpha}{2}$, the perpetual software vendor charges $p_u^{SP2} = (\rho - 1)q$ so that (IC4) is binding. By (IC6), we have $p_n^{SP2} \geq \frac{\alpha}{2} + 2k$. As a result, when $\alpha > 2(\rho - 1)q$, the perpetual software vendor's profit is $\pi_{perp}^{SP2} = (\rho - 1)q$, and the SaaS vendor's profit is $\pi_{SaaS}^{SP2} = \alpha + k - (\rho - \theta)q$.

The optimal prices and profits for $\alpha \leq 2(\rho - 1)q$ are the same as in Proposition 3.

Finally, we compare profits of the two vendors under both SP2 (Upgrade, SaaS) and SP6 (Upgrade+SaaS, New+SaaS). The latter is given in Proposition 4. There are three cases:

Case (1) $(\rho - \theta)q \leq \alpha \leq (\rho + \theta - 2)q$. For the perpetual software vendor, $\pi_{perp}^{SP6} < \pi_{perp}^{SP2}$ if $k < \frac{\alpha(\rho-1)q}{\alpha+(\rho-\theta)q} - \frac{\alpha+(\rho-\theta)q}{4} \doteq k_2$. At both boundary values, $\alpha = (\rho - \theta)q$ and $\alpha = (\rho + \theta - 2)q$, $k_2 = \frac{(\theta-1)q}{2}$. In addition, we can show that there exists $\hat{\alpha} = [2\sqrt{(\rho - 1)(\rho - \theta)} - (\rho - \theta)]q \in [(\rho - \theta)q, (\rho + \theta - 2)q]$ such that $\frac{\partial k_2}{\partial \alpha} > 0$ for $\alpha \in [(\rho - \theta)q, \hat{\alpha}]$ and $\frac{\partial k_2}{\partial \alpha} < 0$ for $\alpha \in [\hat{\alpha}, (\rho + \theta - 2)q]$. Hence, the perpetual software vendor prefers SP2 if $k < k_2$. For the SaaS vendor, $\pi_{SaaS}^{SP6} < \pi_{SaaS}^{SP2}$ if $k > \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha} - (\theta - 1)q - \frac{\alpha}{2} \doteq k_1$. At $\alpha = (\rho - \theta)q$, $k_1 = -(\theta - 1)q - \frac{(\rho-\theta)q}{2} < 0$, and $\frac{\partial k_1}{\partial \alpha} < 0$. Therefore, the inequality always holds. The SaaS vendor always prefers SP2.

Case (2) $(\rho + \theta - 2)q \leq \alpha \leq 2(\rho - 1)q$. For the perpetual software vendor, $\pi_{perp}^{SP6} < \pi_{perp}^{SP2}$ if $k < \frac{8\alpha(\rho-1)q+[\alpha-(\rho+\theta-2)q]^2}{8[\alpha+(\rho-\theta)q]} - \frac{\alpha+(\rho-\theta)q}{4} \doteq k_3$. At $\alpha = (\rho + \theta - 2)q$, $k_3 = \frac{(\theta-1)q}{2}$. Solving $k_3 = 0$, we get two roots. One is smaller than the lower bound $(\rho + \theta - 2)q$, and the other, $\overline{\alpha} = [(\rho + \theta - 2) + 2\sqrt{(\rho - 1)(\rho - \theta)}]q$, is greater than the upper bound $2(\rho - 1)q$. So $k_3 > 0$ in this range and the perpetual software vendor prefers SP2 if $k < k_3$. For SaaS, the condition is the same as in Case (1). The SaaS vendor always prefers SP2.

Case (3) $\alpha > 2(\rho - 1)q$. For the perpetual software vendor, $\pi_{perp}^{SP6} < \pi_{perp}^{SP2}$ if $k < k_3$. The analysis is the same as in Case (2). For the SaaS vendor, $\pi_{SaaS}^{SP6} < \pi_{SaaS}^{SP2}$ if $k > \frac{-[\alpha-(\rho-\theta)q][\alpha+(\rho-\theta)q]}{2\alpha} \doteq k_4$ and $k_4 < 0$. So the SaaS vendor always prefers SP2.

Overall, define $K_2 = \begin{cases} k_2 & if\ \alpha \leq (\rho + \theta - 2)q \\ k_3 & if\ \alpha > (\rho + \theta - 2)q \end{cases}$ and we get the results in Proposition 5.

Appendix F

Effect of α and k—Comparative Statics and Graphical Illustration

In this Appendix, we show how the two key parameters, α and k, affect equilibrium prices, profits, consumer surplus, and social welfare using comparative statics, and we also provide a graphical illustration.

Table F1. Comparative Statistics w.r.t. α

Equilibrium	p_u	p_n	p_s	π_{perp}	π_{SaaS}	CS_{OG}	CS_{NG}	SW
Monopoly (M)	—	—	NA	—	NA	—	—	—
Entry Deterrence (I)	↓	↓	—	↓	—	↑	↑	—
Market Segmentation (IIa)	—	—	↑	—	↑	—	—	↑
Market Segmentation (IIb)	—	—	↑	—	↑	—	↓	↑
Sequential Dominance (IIIa)	↑↓	↑↓	↑	↑↓	↑	↓↑	↑	↑
Sequential Dominance (IIIb)	↓	↑↓	↑	↑↓	↑	↓↑	↑	↑

Table F2. Comparative Statistics w.r.t. *k*

Equilibrium	p_u	p_n	p_s	π_{perp}	π_{SaaS}	CS_{OG}	CS_{NG}	SW
Monopoly (M)	↑	↑	NA	↑	NA	↑	—	↑
Entry Deterrence (I)	↑	↑	0	↑	—	↑	↑	↑
Market Segmentation (IIa)	—	↑	↑	—	↑	↑	—	↑
Market Segmentation (IIb)	—	↑	↑	—	↑	↑	—	↑
Sequential Dominance (IIIa)	↑	↑	—	↑	—	↑	↑	↑
Sequential Dominance (IIIb)	↑	↑	—	↑	—	↑	↑	↑

The graphic demonstrations in Figures F1 and F2 take the following parameter values: $q = 1$, $\rho = 2$, $\theta = 1.2$, and $k = 0.02$. In addition, $\alpha = 0.64$ indicates the equilibrium transition from entry deterrence to market segmentation; $\alpha = 2$ indicates the equilibrium transition from market segmentation II-a to II-b; and $\alpha = 2.25$ indicates the equilibrium transition from market segmentation to sequential dominance.

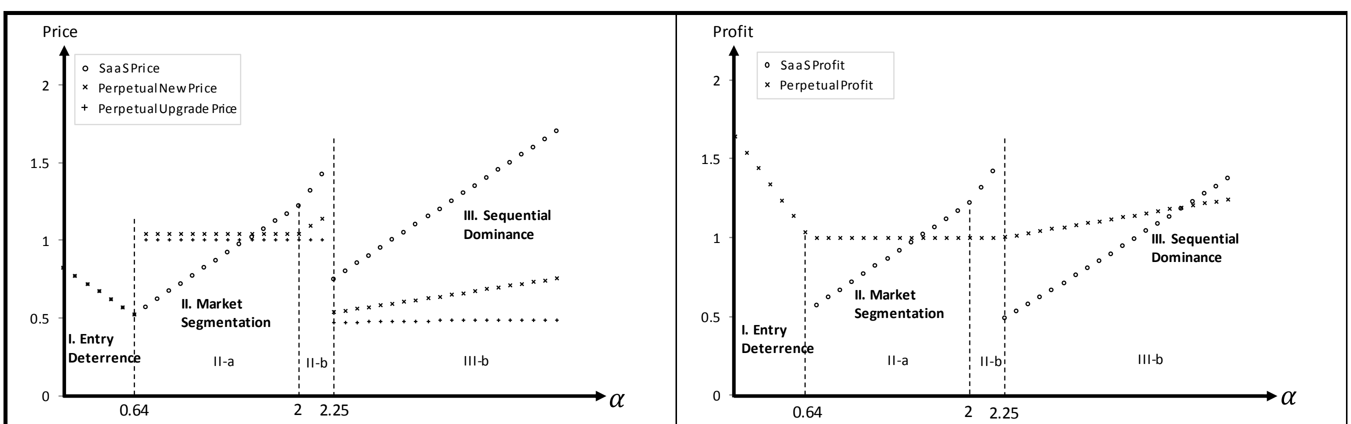

Figure F1. Vendors' Equilibrium Price and Profit Versus SaaS Quality Improvement

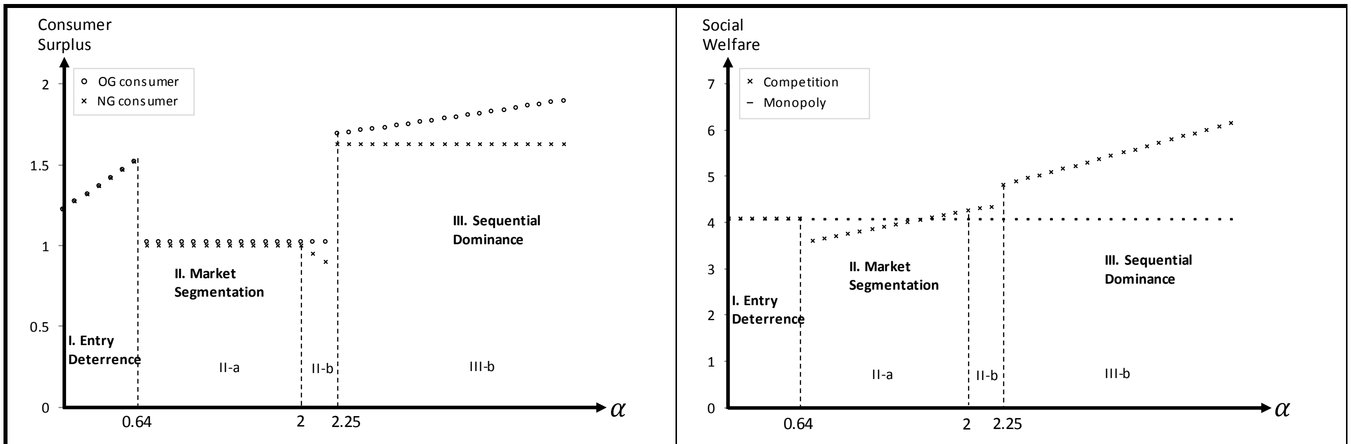

Figure F2. Consumer Surplus and Social Welfare Versus SaaS Quality Improvement

As seen in these figures, when the SaaS's quality improves at a low rate ($\alpha \leq 0.64$), the incumbent perpetual software vendor reduces both upgrade and purchase prices to deter the SaaS vendor's entry, reducing its own profit and resulting in higher consumer surplus. This suggests that the threat of entry by a potential competitor benefits customers.

As α further increases, deterring the SaaS vendor's entry becomes too costly. There is a threshold value ($\alpha = 0.64$) beyond which the perpetual software vendor no longer blocks the SaaS vendor's entry into the market. In the intermediate range of the SaaS quality improvement rate ($0.64 < \alpha \leq 2.25$), the perpetual software vendor pursues the market segmentation strategy by giving up NG users to the SaaS vendor and focusing on serving only OG users with a high price. As a result, its price and profit are independent of the SaaS quality. On the other hand, the SaaS vendor is only interested in exploiting NG users. As the SaaS quality increases at a higher rate, we see that the SaaS's price and profit monotonically increase.

Meanwhile, we observe that consumer surplus for both user groups drops significantly when the perpetual software vendor moves from the entry deterrence to the market segmentation equilibrium after $\alpha = 0.64$. As α increases from 2 to 2.25, the OG users' surplus is unaffected, but surprisingly, the NG users' surplus decreases. The intuition is that, when the SaaS has a large quality advantage over the perpetual software in the range, adopting the perpetual software becomes less attractive to NG users. Therefore, the SaaS vendor is able to price aggressively to extract more consumer surplus from NG users without transferring any benefit to them

Finally, when the SaaS quality improvement rate is high enough ($\alpha > 2.25$), the SaaS becomes very attractive and the perpetual software vendor finds it difficult to prevent OG users from switching to SaaS. Instead, it should reduce both upgrade and purchase prices significantly to compete with the SaaS vendor for both user groups, moving to the sequential dominance strategy. The significant price-reduction pressure from the perpetual software vendor pushes the SaaS vendor to reduce its price as well, which results in a large drop in the SaaS vendor's profit at the transition point ($\alpha = 2.25$). On the other hand, the competition makes users better off, and the consumer surplus for both user groups jumps significantly upward.

As for social welfare, we also observe discrete upward and downward jumps at $\alpha = 0.64$ and 2.25, respectively, when the perpetual software vendor switches its competitive strategy. It is socially inefficient to allow the SaaS vendor to enter the market in the range $0.64 < \alpha < 2$; and after the SaaS vendor enters the market, the resulting social welfare is even lower than the monopoly benchmark. There are two reasons. First, the SaaS software has a low quality in this range. The NG users who adopt the SaaS therefore derive a lower average utility than in the monopoly benchmark, leading to a decrease in social welfare. Second, the SaaS vendor's entry results in a segmented market. Users are not able to enjoy the highest possible network value ($2k$) as they do in the benchmark case. Again, this reduces social welfare.

Appendix G

Perpetual Software Vendor's Incremental Quality Improvement

S1 $(\delta_1, t_{\delta 1})$: Patching before the SaaS Exceeds the Perpetual Software Quality

First, consider SP1 (Upgrade, New). Under SP1, the SaaS vendor is out of the market, even if it prices at 0. To ensure that the OG users prefer Upgrade rather than Old, we need $\rho q + \delta_1 q(1 - t_{\delta 1}) + 2k - p_u \geq q + k$; that is, $p_u \leq (\rho - 1)q + \delta_1 q(1 - t_{\delta 1}) + k$ (G1). To ensure that the OG users prefer Upgrade rather than SaaS, even if SaaS is priced at 0, we need $\rho q + \delta_1 q(1 - t_{\delta 1}) + 2k - p_u \geq \int_0^1 (\theta q + \alpha t + k)dt$; that is, $p_u \leq (\rho - \theta)q + \delta_1 q(1 - t_{\delta 1}) + k - \frac{\alpha}{2}$ (G2). To ensure that the NG users prefer New rather than SaaS, even if SaaS is priced at 0, we must have $\rho q + \delta_1 q(1 - t_{\delta 1}) + 2k - p_n \geq \int_0^1 (\theta q + \alpha t + k)dt$; that is, $p_n \leq (\rho - \theta)q + \delta_1 q(1 - t_{\delta 1}) + k - \frac{\alpha}{2}$ (G3). Therefore, the optimal price is $p_u^{SP1} = p_n^{SP1} = (\rho - \theta)q + \delta_1 q(1 - t_{\delta 1}) + k - \frac{\alpha}{2}$. The optimal profit is $\pi_{perp}^{SP1} = 2(\rho - \theta)q + 2\delta_1 q(1 - t_{\delta 1}) + 2k - \alpha$.

Next, consider SP2 (Upgrade, SaaS). To ensure that the OG users prefer Upgrade rather than Old, we need $\rho q + \delta_1 q(1 - t_{\delta 1}) + k - p_u \geq q + k$; that is, $p_u \leq (\rho - 1)q + \delta_1 q(1 - t_{\delta 1})$ (G4). To ensure that the OG users prefer Upgrade rather than SaaS, we need $\rho q + \delta_1 q(1 - t_{\delta 1}) + k - p_u \geq \int_0^1 (\theta q + \alpha t + 2k - p_s)dt$; that is, $p_u \leq p_s + (\rho - \theta)q + \delta_1 q(1 - t_{\delta 1}) - k - \frac{\alpha}{2}$ (G5). To ensure that the NG users prefer SaaS rather than New, we must have $\int_0^1 (\theta q + \alpha t + k - p_s)dt \geq \rho q + \delta_1 q(1 - t_{\delta 1}) + 2k - p_n$; that is, $p_n \geq p_s + (\rho - \theta)q + \delta_1 q(1 - t_{\delta 1}) + k - \frac{\alpha}{2}$ (G6). To ensure that OG users prefers Upgrade rather than SaaS, we need to make sure that at $t = 1$ the net benefit of switching to SaaS cannot exceed that of Upgrade: $\theta q + \alpha + 2k - p_s \leq (\rho + \delta_1)q + k$; that is, $p_s \geq \alpha + k - (\rho + \delta_1 - \theta)q$ (G7). Therefore, the optimal price is $p_u^{SP2} = (\rho - 1)q + \delta_1 q(1 - t_{\delta 1})$, and the optimal profit is $\pi_{perp}^{SP2} = (\rho - 1)q + \delta_1 q(1 - t_{\delta 1})$. The SaaS price is $p_s^{SP2} = (\theta - 1)q - \delta_1 q(1 - t_{\delta 1}) + k + \frac{\alpha}{2}$ if $\alpha \leq 2(\rho - 1)q - \delta_1 q t_{\delta 1}$; otherwise, $p_s^{SP2} = \alpha + k - (\rho + \delta_1 - \theta)q$.

Comparing the perpetual software vendor's profits under SP1 and SP2, we see that $\pi_{perp}^{SP1} > \pi_{perp}^{SP2}$ if $k > K_1'$, where $K_1' = \frac{\alpha - (\rho - 2\theta + 1)q}{2} - \frac{\delta_1 q(1 - t_{\delta 1})}{2} < K_1$. Consequently, the lower bound value $\underline{\alpha}' = (\rho - 2\theta + 1)q + \delta_1 q(1 - t_{\delta 1}) > \underline{\alpha}$. Both K_1 and $\underline{\alpha}$ are critical values in the baseline model when the perpetual software vendor does not provide a quality jump. Hence, the K_1' line shifts downward and the lower bound $\underline{\alpha}'$ shifts towards right.

Finally, consider SP6 (Upgrade+SaaS, New+SaaS). The switching time is determined by $\theta q + \alpha t_{IQ} + 2k - p_s = (\rho + \delta_1)q + 2k$; that is, $t_{IQ} = \frac{p_s + (\rho + \delta_1 - \theta)q}{\alpha} > t_{s3}$. The SaaS vendor's profit is expressed as $2p_s\left(1 - \frac{p_s + (\rho + \delta_1 - \theta)q}{\alpha}\right)$. Under the condition $\alpha \geq (\rho + \delta_1 - \theta)q$, solving this optimization problem yields the optimal SaaS price $p_s^{SP6} = \frac{\alpha - (\rho + \delta_1 - \theta)q}{2}$, which is lower than the optimal SaaS price under the baseline case.

To ensure that NG users prefer New+SaaS rather than SaaS, we need $(\rho q + 2k)t_{s4} + \delta_1 q(t_{s4} - t_{\delta 1}) - p_n + \int_{t_{s4}}^1 (\theta q + \alpha t + 2k - p_s)dt \geq \int_0^{t_{s4}} (\theta q + \alpha t_s + k - p_s)dt + \int_{t_{s4}}^1 (\theta q + \alpha t + 2k - p_s)dt$. Simplifying this inequality we have $p_n \leq \frac{[\alpha + (\rho + \delta_1 - \theta)q][4k + \alpha + (\rho + \delta_1 - \theta)q]}{8\alpha} - \delta_1 q t_{\delta 1}$ (G8). Furthermore, we need to ensure that OG users prefer Upgrade+SaaS rather than Old+SaaS. The switching time for Old+SaaS is $t_{s1} = \frac{p_s^{SP6} - (\theta - 1)q}{\alpha} = \frac{\alpha - (\rho + \delta_1 + \theta - 2)q}{2\alpha}$. If $\alpha > (\rho + \delta_1 + \theta - 2)q$, then the incentive compatibility condition is $(\rho q + 2k)t_{s4} + \delta_1 q(t_{s4} - t_{\delta 1}) - p_u + \int_{t_{s4}}^1 (\theta q + \alpha t + 2k - p_s)dt \geq (q + k)t_{s1} + \int_{t_{s1}}^{t_{s4}} (\theta q + \alpha t + k - p_s)dt + \int_{t_{s4}}^1 (\theta q + \alpha t + 2k - p_s)dt$. Simplifying this inequality, we have: $p_u \leq \frac{k\alpha - (\rho + \delta_1 - 1)(\theta - 1)q^2 + [\alpha(\rho + \delta_1 - 1) + k(\rho + \delta_1 - \theta)]q}{2\alpha} - \delta_1 q t_{\delta 1}$ (G9). If $\alpha \leq (\rho + \delta_1 + \theta - 2)q$, we need to ensure that OG users prefer Upgrade+SaaS rather than SaaS. Hence, $(\rho q + 2k)t_{s4} + \delta_1 q(t_{s4} - t_{\delta 1}) - p_u + \int_{t_{s4}}^1 (\theta q + \alpha t + 2k - p_s)dt \geq \int_0^{t_{s4}} (\theta q + \alpha t + k - p_s)dt + \int_{t_{s4}}^1 (\theta q + \alpha t + 2k - p_s)dt$, which leads to $p_u \leq \frac{[\alpha + (\rho + \delta_1 - \theta)q][4k + \alpha + (\rho + \delta_1 - \theta)q]}{8\alpha} - \delta_1 q t_{\delta 1}$ (G10). Therefore, $p_u^{SP6} = \frac{k\alpha - (\rho + \delta_1 - 1)(\theta - 1)q^2 + [\alpha(\rho + \delta_1 - 1) + k(\rho + \delta_1 - \theta)]q}{2\alpha} - \delta_1 q t_{\delta 1}$ and $p_n^{SP6} = \frac{[\alpha + (\rho + \delta_1 - \theta)q][4k + \alpha + (\rho + \delta_1 - \theta)q]}{8\alpha} - \delta_1 q t_{\delta 1}$ if $\alpha > (\rho + \delta_1 + \theta - 2)q$; and $p_u^{SP6} = p_n^{SP6} = \frac{[\alpha + (\rho + \delta_1 - \theta)q][4k + \alpha + (\rho + \delta_1 - \theta)q]}{8\alpha} - \delta_1 q t_{\delta 1}$ if $\alpha \leq (\rho + \delta_1 + \theta - 2)q$.

Next, we compare the perpetual software vendor's profits under SP2 and SP6. We find that, compared to the K_2 curve in the baseline model, the new K_2' curve shifts downward. Specifically, if we redefine $\rho' = \rho + \delta_1$, we can write $K_2' = \frac{\alpha(\rho' - 1)q}{\alpha + (\rho' - \theta)q} - \frac{\alpha + (\rho' - \theta)q}{4} + \delta_1 q t_{\delta 1}$ if $\alpha \leq (\rho' +$

$\theta-2)q$ and $K_2'=\frac{8\alpha(\rho'-1)q+[\alpha-(\rho'+\theta-2)q]^2}{8[\alpha+(\rho'-\theta)q]}-\frac{\alpha+(\rho'-\theta)q}{4}+\delta_1 q t_{\delta 1}$ if $\alpha>(\rho'+\theta-2)q$. Compared with K_2, the K_2' curve shifts towards the right. The upper bound $\overline{\alpha}_{S1}'$ is given by $K_2'=0$.

S2 $(\delta_2, t_{\delta 2})$: Patching After the SaaS Exceeds the Perpetual Software Quality

First, consider SP1 (Upgrade, New). The analysis is the same as above. We obtain the same three conditions (G1), (G2), and (G3). So, the solution is also the same: the optimal price is $p_u^{SP1}=p_n^{SP1}=(\rho-\theta)q+\delta_2 q(1-t_{\delta 2})+k-\frac{\alpha}{2}$, and the optimal profit is $\pi_{perp}^{SP1}=2(\rho-\theta)q+2\delta_2 q(1-t_{\delta 2})+2k-\alpha$.

Next, consider SP2 (Upgrade, SaaS). Following the same analysis, we get the same conditions (G4), (G5), and (G6). In addition, we need to ensure that OG users prefer Upgrade rather than Upgrade+SaaS. If OG users chooses to switch from the upgraded perpetual software to SaaS, it must be at $t^*=\frac{(\rho-\theta)q+p_s-k}{\alpha}$. Note that at t^*, the perpetual vendor has not patched its product yet. To ensure that OG users stay with the perpetual software, their expected value from not switching, after considering the future quality improvement $\delta_2 q$ at $t_{\delta 2}$ should be higher than the expected value from switching to SaaS: $\int_{t^*}^{t_{\delta 2}}(\theta q+\alpha t+2k-p_s)dt-(\rho q+k)(t_{\delta 2}-t^*)\le(\rho q+\delta_2 q+k)(1-t_{\delta 2})-\int_{t_\delta}^{1}(\theta q+\alpha t+2k-p_s)dt$. Simplifying and solving this inequality yields $p_s\ge\alpha+k-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}$ (G11). Using (G4), we get the optimal upgrade price $p_u^{SP2}=(\rho-1)q+\delta_2 q(1-t_{\delta 2})$. Substituting p_u^{SP2} into (G5), we get $p_s\ge(\theta-1)q+k+\frac{\alpha}{2}$. Now we compare this lower bound of p_s with the condition (G11): Define $\Delta\doteq(\theta-1)q+k+\frac{\alpha}{2}-\left\{\alpha+k-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}\right\}$. When $\alpha<2\delta_2 q(1-t_{\delta 2})$, $\Delta>0$. When $\alpha\ge 2\delta_2 q(1-t_{\delta 2})$, $\Delta_{\alpha=2\delta_2 q(1-t_{\delta 2})}>0$ and $\frac{\partial\Delta}{\partial\alpha}<0$. So if α exceeds a certain threshold value, $\Delta<0$. At the largest possible value of $\alpha_{max}=(\rho+\delta_2-\theta)q$, we find that $\Delta_{\alpha=(\rho+\delta_2-\theta)q}>0$. Therefore, we always have $\Delta>0$. Consequently, the optimal SaaS price is $p_s^{SP2}=(\theta-1)q+k+\frac{\alpha}{2}$, at which the non-switching condition (G11) is always satisfied. The perpetual software prices are $p_u^{SP2}=p_n^{SP2}=(\rho-1)q+\delta_2 q(1-t_{\delta 2})$, and the profit is $\pi_{perp}^{SP2}=(\rho-1)q+\delta_2 q(1-t_{\delta 2})$.

Next, we compare the perpetual software vendor's profits under SP1 and SP2: $\pi_{perp}^{SP1}>\pi_{perp}^{SP2}$ if $k>K_1'$, where $K_1'=\frac{\alpha-(\rho-2\theta+1)q}{2}-\frac{\delta_2 q(1-t_{\delta 2})}{2}$. Note that both the K_1' line and lower bound value $\underline{\alpha}'$ are as same as in the above Patching Strategy S1.

Finally, consider SP6 (Upgrade+SaaS, New+SaaS). The switching time is determined by $\theta q+\alpha t^*+2k-p_s=\rho q+2k$; that is, $t^*=\frac{p_s+(\rho-\theta)q}{\alpha}$. The SaaS vendor's profit is expressed as $2p_s\left(1-\frac{p_s+(\rho-\theta)q}{\alpha}\right)$. It yields the optimal SaaS price $p_s^*=\frac{\alpha-(\rho-\theta)q}{2}$, which is the same as the optimal SaaS price in the baseline model. For SP6 to be an equilibrium, we need to ensure switching does happen. That is, at t^*, it must be $\int_{t^*}^{t_{\delta 2}}(\theta q+\alpha t-p_s)dt-\rho q(t_{\delta 2}-t^*)\ge(\rho q+\delta_2 q)(1-t_{\delta 2})-\int_{t_{\delta 2}}^{1}(\theta q+\alpha t-p_s)dt$. Simplifying and solving this inequality yields $p_s\le\alpha-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}$ (G12). Now we check whether the SaaS price $p_s^*=\frac{\alpha-(\rho-\theta)q}{2}$ from the above optimization problem satisfies (G12). We can show that if $\delta_2 q(1-t_{\delta 2})\le\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$, p_s^* satisfies (G12) and so $p_s^{SP6}=\frac{\alpha-(\rho-\theta)q}{2}$, $t^*=\frac{\alpha+(\rho-\theta)q}{2\alpha}$; otherwise, p_s^* does not satisfy (G12), and so $p_s^{SP6}=\alpha-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}$, $t^*=\frac{\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}}{\alpha}$.

We need to ensure that NG users prefer New+SaaS rather than SaaS. That is, $(\rho q+2k)t^*-p_n+\int_{t^*}^{1}(\theta q+\alpha t+2k-p_s)dt\ge\int_0^{t^*}(\theta q+\alpha t+k-p_s)dt+\int_{t^*}^{1}(\theta q+\alpha t_s+2k-p_s)dt$. When $\delta_2 q(1-t_{\delta 2})\le\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$, the condition leads to $p_n\le\frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$ (G13); otherwise, $p_n\le\frac{[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}][2k+\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]}{2\alpha}$ (G14).

We also need to ensure that OG users prefer Upgrade+SaaS rather than Old+SaaS. The switching time in Old+SaaS is $t_{s1}=\frac{p_s^{SP6}-(\theta-1)q}{\alpha}$. According to different values of $\delta_2 q(1-t_{\delta 2})$, we analyze the following two cases.

Case (a) When $\delta_2 q(1-t_{\delta 2})\le\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$, $t_{s1}=\frac{\alpha-(\rho+\theta-2)q}{2\alpha}$. If $\alpha>(\rho+\theta-2)q$, $t_{s1}>0$, and the incentive compatibility condition is $(\rho q+2k)t^*-p_u+\int_{t^*}^{1}(\theta q+\alpha t+2k-p_s)dt\ge(q+k)t_{s1}+\int_{t_{s1}}^{t^*}(\theta q+\alpha t+k-p_s)dt+\int_{t^*}^{1}(\theta q+\alpha t+2k-p_s)dt$. Simplifying it we have $p_u\le\frac{k\alpha-(\rho-1)(\theta-1)q^2+[\alpha(\rho-1)+k(\rho-\theta)]q}{2\alpha}$ (G15). Hence, the optimal perpetual software prices are given by (G13) and (G15). If $\alpha<(\rho+\theta-2)q$, $t_{s1}<0$, so the incentive compatibility condition is to ensure that OG users prefer Upgrade+SaaS rather than SaaS: $(\rho q+$

$2k)t^* - p_u + \int_{t^*}^{1} (\theta q + \alpha t + 2k - p_s)dt \geq \int_0^{t^*} (\theta q + \alpha t + k - p_s)dt + \int_{t^*}^{1} (\theta q + \alpha t + 2k - p_s)dt$, which leads to $p_u \leq \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$ (G16). Hence, the optimal perpetual software prices are given by (G13) and (G16).

Case (b) When $\delta_2 q(1-t_{\delta 2}) > \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$, $t_{s1} = \frac{\alpha-(\rho-1)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}}{\alpha}$. If $\delta_2 q(1-t_{\delta 2}) < \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$, $t_{s1} > 0$, and the incentive compatibility condition is to ensure that OG users prefer Upgrade+SaaS other than Old+SaaS. Then we have $p_u \leq [(\rho-1)q + k]\frac{\alpha-\sqrt{2\delta_2 q(1-t_{\delta 2})}}{\alpha} - \frac{(\rho-1)^2 q^2}{2\alpha}$ (G17). Hence, the optimal perpetual software prices are given by (G14) and (G17). If $\delta_2 q(1-t_{\delta 2}) > \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$, $t_{s1} < 0$, so the incentive compatibility condition is to ensure that OG users prefer Upgrade+SaaS rather than SaaS. Similarly, we get $p_u \leq \frac{[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}][2k+\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]}{2\alpha}$ (G18). Hence, the optimal perpetual software prices are given by (G14) and (G18).

Note that $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} > \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$ when $\alpha < (\rho+\theta-2)q$, and $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} < \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$ when $\alpha > (\rho+\theta-2)q$. As a result, the optimal prices and vendor profits in SP6 can be summarized in the following, depending on both $\delta_2 q(1-t_{\delta 2})$ and α. Define $\underline{v} = min\left\{\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}, \frac{[\alpha-(\rho-1)q]^2}{2\alpha}\right\}$ and $\overline{v} = max\left\{\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}, \frac{[\alpha-(\rho-1)q]^2}{2\alpha}\right\}$. We have three cases:

(i) $\delta_2 q(1-t_{\delta 2}) < \underline{v}$: if $\alpha < (\rho+\theta-2)q$, $p_s^{SP6} = \frac{\alpha-(\rho-\theta)q}{2}$, $p_u^{SP6} = p_n^{SP^6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$, $\pi_{SaaS}^{SP6} = \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$, and $\pi_{perp}^{SP6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{4\alpha}$; if $\alpha > (\rho+\theta-2)q$, $p_s^{SP6} = \frac{\alpha-(\rho-\theta)q}{2}$, $p_n^{SP6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$, $p_u^{SP^6} = \frac{k\alpha-(\rho-1)(\theta-1)q^2+[\alpha(\rho-1)+k(\rho-\theta)]q}{2\alpha}$, $\pi_{SaaS}^{SP6} = \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$, and $\pi_{perp}^{SP6} = \frac{2[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]-[\alpha-(\rho+\theta-2)q]^2}{8\alpha}$.

(ii) $\underline{v} < \delta_2 q(1-t_{\delta 2}) < \overline{v}$: if $\alpha < (\rho+\theta-2)q$, $p_s^{SP6} = \frac{\alpha-(\rho-\theta)q}{2}$, $p_u^{SP6} = p_n^{SP^6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{8\alpha}$, $\pi_{SaaS}^{SP6} = \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$, $\pi_{perp}^{SP6} = \frac{[\alpha+(\rho-\theta)q][4k+\alpha+(\rho-\theta)q]}{4\alpha}$; if $\alpha > (\rho+\theta-2)q$, $p_s^{SP6} = \alpha-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}$, $p_n^{SP6} = \frac{[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}][2k+\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]}{2\alpha}$, $p_u^{SP6} = [(\rho-1)q+k]\frac{\alpha-\sqrt{2\delta_2 q(1-t_{\delta 2})}}{\alpha} - \frac{(\rho-1)^2 q^2}{2\alpha}$, $\pi_{SaaS}^{SP6} = \frac{2\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}[\alpha-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]^2}{\alpha}$, $\pi_{perp}^{SP6} = \frac{\left[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}\right]^2+2[(\rho-1)q+2k](\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}-[(\rho-1)q]^2}{2\alpha}$.

(iii) $\delta_2 q(1-t_{\delta 2}) > \overline{v}$: $p_n^{SP6} = p_u^{SP6} = \frac{[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}][2k+\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]}{2\alpha}$, $p_s^{SP6} = \alpha-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}$, $\pi_{SaaS}^{SP6} = \frac{2\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}[\alpha-(\rho-\theta)q-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]^2}{\alpha}$, and $\pi_{perp}^{SP6} = \frac{[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}][2k+\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]}{\alpha}$.

Finally, we compare the perpetual software vendor's profits under SP2 and SP6. The comparison should be done in each region of $\delta_2 q(1-t_{\delta 2})$. In (i), when $\delta_2 q(1-t_{\delta 2})$ is small, the perpetual vendor's profit in SP6, π_{perp}^{SP6}, is the same as in the baseline model. Hence, the $K_2' = K_2 + \frac{\alpha}{\alpha+(\rho-\theta)q}\delta_2 q(1-t_{\delta 2})$ curve that divides the market segmentation equilibrium (SP2) and the sequential dominance equilibrium (SP6) shifts upward and toward the right, compared to the K_2 curve in the baseline model. Similarly, in (ii), we have $K_2' = \frac{\alpha(\rho-1)q}{\alpha+(\rho-\theta)q} - \frac{\alpha+(\rho-\theta)q}{4} + \frac{\alpha}{\alpha+(\rho-\theta)q}\delta_2 q(1-t_{\delta 2})$ if $\alpha < (\rho+\theta-2)q$ and $K_2' = \frac{\alpha[(\rho-1)q+\delta_2 q(1-t_{\delta 2})]}{2[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]} + \frac{[(\rho-1)q]^2-\left[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}\right]^2}{4\left[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}\right]} - \frac{(\rho-1)q}{2}$ if $\alpha \geq (\rho+\theta-2)q$. In (iii), we have $K_2' = \frac{\alpha[(\rho-1)q+\delta_2 q(1-t_{\delta 2})]}{2[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]} - \frac{[\alpha-\sqrt{2\alpha\delta_2 q(1-t_{\delta 2})}]}{2}$. Under the three cases, the upper bound $\overline{\alpha}'_{S2(i)}$, $\overline{\alpha}'_{S2(ii)}$ and $\overline{\alpha}'_{S2(iii)}$ are given by solving $K_2' = 0$. Furthermore, $\overline{\alpha}'_{S2(iii)} > \overline{\alpha}'_{S2(ii)} > \overline{\alpha}'_{S2(i)}$, and $\overline{\alpha}'_{S2(i)} > \overline{\alpha}$.

To conclude, in each case, there are no qualitative changes in the competition outcomes, except that the equilibrium regions are shifted.

Proof of Proposition 6 (Optimal Patching Strategy and Time)

We show the proof based on a special case $k = 0$. The reasoning for the general case is similar. We omit the proof because the mathematical expressions are quite lengthy.

Define $\alpha_1 = \overline{\alpha}'_{S1}$ and $\alpha_2 = \overline{\alpha}'_{S2(i)}$ where $\overline{\alpha}'_{S1}$ and $\overline{\alpha}'_{S2(i)}$ are the upper bound in S1 and S2, respectively. When $\alpha < \alpha_1$, the equilibrium under S1 and S2 is the same (either entry deterrence or market segmentation). The perpetual software vendor's profit functions are also the same. Since its profit is linearly increasing in the patching value, the optimal patching time is determined by solving the largest patching value: $t_\delta^* = \underset{\forall t_\delta \in (0,1)}{Argmax}\{\delta q(1-t_\delta)\}$. It can be either before or after t^*.

When $\alpha_1 < \alpha < \alpha_2$, for any patching value, the equilibrium under S1 is sequential dominance and under S2 is market segmentation. Next we compare the two equilibrium profits for the perpetual software vendor. Define $v_1 \equiv \frac{[\alpha-(\rho+\delta_1-\theta)q]^2}{8\alpha} + \frac{(\rho+\delta_1-1)[\alpha-(\theta-1)]q^2}{2\alpha} - (\rho-1)q - 2\delta_1 q t_{\delta 1}$. If $V_{S2} > v_1$, S2 offers a higher profit than S1. The vendor's profit π_{perp}^{SP2} under S2 is linearly increasing in its patching value. The optimal patching time is given by $t_{\delta 2}^* = \underset{\forall t_\delta \in (t^*,1)}{Argmax}\{\delta q(1-t_\delta)\}$. So the optimal patching time should be later than t^*. If $V_{S2} < v_1$, S1 offers a higher profit than S2, and the optimal patching time should be earlier than t^*. The optimal patching time is determined by solving the profit maximization problem under π_{perp}^{SP6}: $\underset{t_\delta \in (0,t^*)}{Max}\left\{\frac{[\alpha+(\rho+\delta t_\delta-\theta)q]^2}{8\alpha} + \frac{(\rho+\delta t_\delta-1)q[\alpha-(\theta-1)q]}{2\alpha} - 2\delta q t_\delta\right\}$.

When $\alpha > \alpha_2$, the equilibrium under S1 is sequential dominance. Consider two possibilities. (1) If $V_{S2} < \underline{v}$, the equilibrium under S2 is sequential dominance as in the aforementioned case (i). The perpetual software vendor's profit π_{perp}^{SP6} under S2 is the same as in the baseline model. It does not depend on the patching value V_{S2} at all. So it is always smaller than the profit π_{perp}^{SP6} under S1. The vendor therefore should prefer S1, and its optimal patching time should be earlier than t^* and it maximizes π_{perp}^{SP6} under S1: $\underset{t_\delta \in (0,t^*)}{Max}\left\{\frac{[\alpha+(\rho+\delta t_\delta-\theta)q]^2}{8\alpha} + \frac{(\rho+\delta t_\delta-1)q[\alpha-(\theta-1)q]}{2\alpha} - 2\delta q t_\delta\right\}$. (2) If $V_{S2} > \underline{v}$, under S2, we are in cases (ii) and (iii). However, $\overline{\alpha}'_{S2(iii)} > \overline{\alpha}'_{S2(ii)} > (\rho+\delta_2-\theta)q$. The resulting equilibrium is market segmentation. Hence, we compare π_{perp}^{SP6} under S1 and π_{perp}^{SP2} under S2. The analysis and results are the same as those in $\alpha_1 < \alpha < \alpha_2$: If $V_{S2} < v_1$, the optimal patching time should be before t^*; otherwise, the optimal patching time should be after t^*

Define $v_2 \equiv max(v_1, \underline{v})$. By combining the above analyses in all regions of α and V_{S2}, we complete the proof of Proposition 6.

Appendix H

Perpetual Software Vendor's Major Quality Improvement (Two-Period Model)

When $\alpha \leq (\rho-\theta)q$, the SaaS quality improvement rate is small such that the perpetual software always has the quality advantage in both periods. In this case, the perpetual software vendor can deter SaaS entry. The corresponding equilibrium strategy pair is SP1$'$[(Upgrade1, Upgrade2), (New1, Upgrade2)].

When $\alpha > (\rho-\theta)q$, the SaaS entry cannot be deterred. There are two cases. If $(\rho-\theta)q < \alpha \leq (\rho-1)q$, the single-period quality improvement of SaaS is smaller than that of the perpetual software. Because the SaaS has relative quality advantage in the first period but not in the second period, the possible equilibrium strategies are either SP3$'$[(Upgrade1+SaaS, Upgrade2+SaaS), (New1+SaaS, Upgrade2+SaaS)] or SP3$''$[(Upgrade1+SaaS, Upgrade2), (New1+SaaS, Upgrade2)].

If $(\rho-1)q < \alpha \leq (2\rho-\theta-1)q)$, the single-period quality improvement of SaaS is larger than that of the perpetual software. Because the SaaS has relative quality advantage in the second period but not in the first period, the possible strategies are either SP3$'$[(Upgrade1+SaaS, Upgrade2+SaaS), (New1+SaaS, Upgrade2+SaaS)] or SP3$'''$[(Upgrade1, Upgrade2+SaaS), (New1, Upgrade2+SaaS)].

Furthermore, because the perpetual software has quality advantage at the beginning of each period, and it has OG users as the established customer base, the perpetual software vendor might consider the market segmentation strategy to give up the NG users in both periods or only in one period. The possible equilibrium strategies are SP2$'$[(Upgrade1, Upgrade2), (SaaS, SaaS)] for all α, SP2$''$[(Upgrade1, Upgrade2), (SaaS, New2)] if $(\rho-\theta)q < \alpha \leq (\rho-1)q$. Note that if $(\rho-1)q < \alpha \leq (2\rho-\theta-1)q)$, SP2$'''$[(Upgrade1, Upgrade2), (New1, SaaS)] cannot emerge as equilibrium because after OG users upgrade and NG users adopt the new perpetual software, their actions should be the same.

Entry Deterrence Strategy

Consider SP1$'$[(Upgrade1, Upgrade2), (New1, Upgrade2)]. Because the SaaS vendor can reduce price to zero, to prevent users from switching to SaaS at anytime between [0,2], we need $\theta q + \alpha \leq \rho q$; that is, $\alpha \leq (\rho-\theta)q$.

Given that the NG users adopt the perpetual software in both periods, to ensure that the OG users prefer upgrading in both periods rather than just in the first period, we have $\rho q + 2k + (2\rho-1)q + 2k - 2p_u \geq \rho q + 2k + \rho q + k - p_u$; that is, $p_u \leq (\rho-1)q + k$ (H1). Similarly, given that the OG users choose to upgrade in both periods, to ensure that the NG users prefer to buy new perpetual software and upgrade in period 2 rather than not upgrading, their total utility must be $\rho q + 2k + (2\rho-1)q + 2k - p_n - p_u \geq \rho q + 2k + \rho q + k - p_n$, which is the same as (H1).

To ensure that OG users prefer upgrading in both periods rather than adopting SaaS in any period, even if the SaaS price is reduced to zero, the entry deterrence condition is $(\rho q+2k)+(2\rho-1)q+2k-2p_u \geq max[\int_0^2 (\theta q+\alpha t+k)dt, \int_0^1 (\theta q+\alpha t+k)dt+(2\rho-1)q+2k-p_u, \int_1^2 (\theta q+\alpha t+k)dt+\rho q+2k-p_u]$. In addition, to ensure that the NG users prefer (New1, Upgrade2) to the SaaS in any period, even if the SaaS price is zero, their total utility must be $\rho q+2k+(2\rho-1)q+2k-p_n-p_u \geq max[\int_0^2 (\theta q+\alpha t+k)dt, \int_0^1 (\theta q+\alpha t+k)dt+(2\rho-1)q+2k-p_n, \int_1^2 (\theta q+\alpha t+k)dt+\rho q+2k-p_n]$. Solving these inequalities, we have $p_u \leq (\rho-\theta)q+k-\frac{\alpha}{2}$ (H2) and $p_n+p_u \leq (3\rho-2\theta-1)q+2k-2\alpha$ (H3).

Comparing (H1) and (H2) we see (H1) is not binding. So by (H2) the perpetual software vendor sets the upgrade price at the upper bound $p_u=(\rho-\theta)q+k-\frac{\alpha}{2}$, and by (H3) $p_n=(2\rho-\theta-1)q+k-\frac{3\alpha}{2}$. We can verify that $p_u<p_n$. Consequently, the perpetual software vendor' s profit is $\pi_{perp}^{SP1\prime}=3p_u+p_n=(5\rho-4\theta-1)q+4k-3\alpha$, and the SaaS vendor is out of the market.

Market Segmentation Strategy

Case (1) Consider SP2′[(Upgrade1, Upgrade2), (SaaS, SaaS)]. To prevent the OG users from switching to SaaS, the SaaS payoff at the end of each period should not be higher than payoff from the new perpetual software for OG users. Thus, we have $\theta q+\alpha+2k-p_s \leq \rho q+k$, and $\theta q+2\alpha+2k-p_s \leq (2\rho-1)q+k$. Hence, if $\alpha \leq (\rho-1)q$, $p_s \geq \alpha+k-(\rho-\theta)q$ (H4); and if $\alpha>(\rho-1)q$, $p_s \geq 2\alpha+k-(2\rho-\theta-1)q$ (H5).

Given that the NG users adopt SaaS in both periods, to ensure that the OG users prefer to upgrade in both periods rather than opt for SaaS, their total utility must be $\rho q+k+(2\rho-1)q+k-2p_u \geq \int_0^2 (\theta q+\alpha t+2k-p_s)dt$ and thus $p_u \leq p_s+\frac{(3\rho-2\theta-1)q}{2}-k-\alpha$ (H6). To ensure the OG users to upgrade in both periods rather than just in one period, we must have $\rho q+k+(2\rho-1)q+k-2p_u \geq max[2(\rho q+k)-p_u, q+k+(2\rho-1)q+k-p_u]$; that is, $p_u \leq (\rho-1)q$ (H7).

Similarly, given that the OG users upgrade in both periods, to ensure that the NG users prefer (SaaS, SaaS) rather than (SaaS, New2), we must have $\int_0^2 (\theta q+\alpha t+k-p_s)dt \geq \int_0^1 (\theta q+\alpha t+k-p_s)dt+(2\rho-1)q+2k-p_n$; which is $p_n \geq p_s+(2\rho-\theta-1)q+k-\frac{3\alpha}{2}$ (H8). To ensure that the NG users prefer (SaaS, SaaS) rather than (New1, Upgrade2), we must have $\int_0^2 (\theta q+\alpha t+k-p_s)dt \geq \rho q+2k+(2\rho-1)q+2k-p_n-p_u$; that is, $p_n+p_u \geq 2p_s+(3\rho-2\theta-1)q+2k-2\alpha$ (H9).

If $\alpha \leq (\rho-1)q$, to maximize its profit, the perpetual software vendor charges $p_u=(\rho-1)q$ and sets p_n high enough such that the SaaS vendor can charge a high enough price p_s, so that the OG users would not opt for SaaS. By binding constraint (H6), we have $p_s=\frac{(2\theta-\rho-1)q}{2}+k+\alpha$. We can verify that (H4) is satisfied. By (H8) and (H9), $p_n=max[\frac{3(\rho-1)q}{2}+2k-\frac{3\alpha}{2}, 2(\rho-1)q+4k]$. The perpetual software vendor's profit is $\pi_{perp}^{SP2\prime}=2(\rho-1)q$, and the SaaS vendor's profit is $\pi_{saas}^{SP2\prime}=(2\theta-\rho-1)q+2k+2\alpha$.

If $(\rho-1)q<\alpha \leq \frac{3(\rho-1)q}{2}$, (H5) can be satisfied and the same solution as above holds.

If $\alpha>\frac{3(\rho-1)q}{2}$, then we obtain the boundary solution $p_s=2\alpha+k-(2\rho-\theta-1)q$. Now, (H8) becomes $p_n \geq 2k+\frac{\alpha}{2}$, and (H9) becomes $p_n+p_u \geq 4k+2\alpha-(\rho-1)q$. So $p_u=(\rho-1)q$ and $p_n=4k+2\alpha-2(\rho-1)q$. The perpetual software vendor's profit is $\pi_{perp}^{SP2\prime}=2(\rho-1)q$, and the SaaS vendor's profit is $\pi_{saas}^{SP2\prime}=4\alpha+2k-2(2\rho-\theta-1)q$.

Comparing $\pi_{perp}^{SP2\prime}$ with $\pi_{perp}^{SP1\prime}$ we see that if $k>\frac{3\alpha-(3\rho-4\theta+1)q}{4}=K_1'$, then $\pi_{perp}^{SP1\prime}>\pi_{perp}^{SP2\prime}$, the entry deterrence strategy dominates the market segmentation strategy. Solving $K_1'=0$ we get $\underline{\alpha}'$.

Case (2) If $(\rho-\theta)q<\alpha \leq (\rho-1)q$, consider SP2″[(Upgrade1, Upgrade2), (SaaS, New2)]. Given that the NG users adopt (SaaS, New2), OG users prefer (Upgrade1, Upgrade2) rather than (SaaS, Upgrade2) if $\rho q+k-\mathrm{p}_u \geq \int_0^1 (\theta q+\alpha t+2k-p_s)dt$; that is $p_u \leq p_s+(\rho-\theta)q-k-\frac{\alpha}{2}$ (H10). Given that OG users upgrade in both periods, to ensure NG users prefer (SaaS, New2) rather than (New1, Upgrade2), we need $\int_0^1 (\theta q+\alpha t+k-p_s)dt+(2\rho-1)q+2k-p_n \geq \rho q+2k-p_n+(2\rho-1)q+2k-p_u$; that is, $p_u \geq p_s+(\rho-\theta)q+k-\frac{\alpha}{2}$ (H11). Because (H10) and (H11) contradict with each other, this user strategy does not support an equilibrium.

Sequential Dominance Strategy

When $\alpha \geq (\rho-\theta)q$, the two competing firms' periodical quality improvement is competitive against each other. There are three possible strategies:

(1) SP3′[(Upgrade1+SaaS, Upgrade2+SaaS), (New1+SaaS, Upgrade2+SaaS)]. This symmetric strategy can occur in both $\alpha \leq (\rho-1)q$ and $\alpha > (\rho-1)q$ ranges.

(2) SP3″[(Upgrade1+SaaS, Upgrade2), (New1+SaaS, Upgrade2)]. This asymmetric strategy can only occur when $\alpha \leq (\rho-1)q$; that is, the perpetual software vendor has higher single-period quality improvement than the SaaS vendor.

(3) SP3‴[(Upgrade1, Upgrade2+SaaS), (New1, Upgrade2+SaaS)]. This asymmetric strategy can only occur when $\alpha > (\rho-1)q$; that is, the SaaS has higher single-period quality improvement than the perpetual software.

Case (1) Consider SP3′. The sequential dominance strategy involves user switching. If users switch from the new/updated perpetual software to SaaS in the first period, the switching time is determined by $\theta q + \alpha t_{\sigma 1} + 2k - p_s = \rho q + 2k$; that is, $t_{\sigma 1} = \frac{p_s + (\rho-\theta)q}{\alpha}$. If users switch from the updated perpetual software to SaaS in the second period, the switching time is determined by $\theta q + \alpha t_{\sigma 2} + 2k - p_s = (2\rho-1)q + 2k$; that is, $t_{\sigma 2} = \frac{p_s + (2\rho-\theta-1)q}{\alpha}$. If users switch from the old version software to SaaS, the switching time is determined by $\theta q + \alpha t_{\sigma 3} + k - p_s = q + k$, so that $t_{\sigma 3} = \frac{p_s - (\theta-1)q}{\alpha}$.

If the SaaS vendor would like to serve in both periods, we need $0 < t_{\sigma 1} < 1$ and $1 < t_{\sigma 2} < 2$. That is, if $\alpha \leq (\rho-1)q$, $\alpha - (2\rho-\theta-1)q < p_s \leq 2\alpha - (2\rho-\theta-1)q$ (H12); if $\alpha > (\rho-1)q$, $\alpha - (2\rho-\theta-1)q < p_s \leq \alpha - (\rho-\theta)q$ (H13). The SaaS vendor's profit is $2p_s(1-t_{\sigma 1}) + 2p_s(2-t_{\sigma 2})$. Solving this optimization problem we have interior solution $p_s^* = \frac{3\alpha - (3\rho-2\theta-1)q}{4}$. Checking (H12) and (H13) we can verify that this interior solution holds if $\frac{(5\rho-2\theta-3)q}{5} < \alpha < (5\rho-2\theta-3)q$.

At this interior solution, given that the OG users choose (Upgrade1+SaaS, Upgrade2+SaaS), in order for NG users to prefer (New1+SaaS, Upgrade2+SaaS) rather than (SaaS, New2+SaaS), we need $(\rho q + 2k)t_{\sigma 1} - p_u \geq \int_0^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)dt$, which is $p_u \leq \frac{[p_s + (\rho-\theta)q + 2k][p_s + (\rho-\theta)q]}{2\alpha}$ (H14). In order for NG users to prefer (New1+SaaS, Upgrade2+SaaS) rather than (SaaS, SaaS), we have $(\rho q + 2k)t_{\sigma 1} - p_n + [(2\rho-1)q + 2k](t_{\sigma 2} - 1) - p_u \geq \int_0^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)dt + \int_1^{t_{\sigma 2}} (\theta q + \alpha t + k - p_s)dt$; that is, $p_n + p_u \leq \frac{[p_s + (\rho-\theta)q + 2k][p_s + (\rho-\theta)q] + [p_s + (2\rho-\theta-1)q - \alpha + 2k][p_s + (2\rho-\theta-1)q - \alpha]}{2\alpha}$ (H15). Given the NG users choose (New1+SaaS, Upgrade2+SaaS), in order for the OG users to prefer (Upgrade1+SaaS, Upgrade2+SaaS) rather than (Old+SaaS, Upgrade2+SaaS), we need $(\rho q + 2k)t_{\sigma 1} - p_u \geq (q+k)t_{\sigma 3} + \int_{t_{\sigma 3}}^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)dt$. Solving this inequality we have $p_u \leq \frac{2[(\rho-1)q + k]p_s + 2k(\rho-\theta)q + (\rho-1)(\rho-2\theta+1)q^2}{2\alpha}$ (H16).

If $\alpha \leq \frac{(3\rho+2\theta-5)q}{3}$, $t_{\sigma 3} \leq 0$. In order for the OG users to prefer (Upgrade1+SaaS, Upgrade2+SaaS) rather than (SaaS, Upgrade2+SaaS), we need $(\rho q + 2k)t_{\sigma 1} - p_u \geq \int_0^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)dt$, which is the same as (H14). If $\alpha > \frac{(3\rho+2\theta-5)q}{3}$, $t_{\sigma 3} \geq 0$. Comparing (H14) and (H16) we can verify that (H16) binds. Therefore, for the SP3′ interior solution, we have the following:

If $\frac{(5\rho-2\theta-3)q}{5} < \alpha \leq (\rho-1)q$, (H14) binds. So $p_u = \frac{[(\rho-2\theta+1)q + 3\alpha + 8k][(\rho-2\theta+1)q + 3\alpha]}{32\alpha}$ and $p_n = \frac{[(5\rho-2\theta-3)q - \alpha + 8k][(5\rho-2\theta-3)q - \alpha]}{32\alpha}$. Furthermore, $p_u < p_n$.

If $(\rho-1)q < \alpha \leq \frac{(3\rho+2\theta-5)q}{3}$, (H14) binds. So we have $p_u = p_n = \frac{5\alpha^2 + 8\alpha k + (24k\rho - 16k\theta - 2\alpha\rho - 4\alpha\theta + 6\alpha - 8k)q + (13\rho^2 - 12\rho\theta + 4\theta^2 - 14\rho + 4\theta + 5)q^2}{32\alpha}$.

If $\frac{(3\rho+2\theta-5)q}{3} < \alpha < (2\rho-\theta-1)q$, (H16) imposes an upper bound for p_u. If $k > k_1 = \frac{21\rho^2q^2 + 4\rho\theta q^2 + 4\theta^2q^2 - 26\alpha\rho q - 4\alpha\theta q - 46\rho q^2 - 12\theta q^2 + 5\alpha^2 + 30\alpha q + 29q^2}{16[\alpha - (\rho-1)q]}$, we still have $p_u = p_n = \frac{5\alpha^2 + 8\alpha k + (24k\rho - 16k\theta - 2\alpha\rho - 4\alpha\theta + 6\alpha - 8k)q + (13\rho^2 - 12\rho\theta + 4\theta^2 - 14\rho + 4\theta + 5)q^2}{32\alpha}$. We can verify that the condition $k > k_1$ always holds in this α range.

Now consider the boundary solution. If $(\rho-\theta)q \leq \alpha \leq \frac{(5\rho-2\theta-3)q}{5}$, then the SaaS vendor prices at boundary solution $p_s^* = 2\alpha - (2\rho-\theta-1)q$. Correspondingly, $t_{\sigma 2} = 2$. SP3′ degenerates to equilibrium SP3″[(Upgrade1+SaaS, Upgrade2), (New1+SaaS, Upgrade2)]. Substituting p_s^* into (H14) we have $p_u = \frac{[2\alpha - (\rho-1)q + 2k][2\alpha - (\rho-1)q]}{2\alpha}$. By (H15) we have $p_u = k + \frac{\alpha}{2}$.

If $\alpha > (5\rho - 2\theta - 3)q$, then the SaaS vendor prices at boundary price $p_s^* = \alpha - (\rho - \theta)q$. Correspondingly, $t_{\sigma 1} = 1$. SP3$'$ degenerates to equilibrium SP3$'''$[(Upgrade1, Upgrade2+SaaS), (New1, Upgrade2+SaaS)]. However, note that $(5\rho - 2\theta - 3)q > (2\rho - \theta - 1)q$. So the degenerated SP3$'''$ does not occur in the α range we consider.

Case (2) Consider SP3$''$. Knowing it only serves in one period, the SaaS vendor's optimization problem becomes $2p_s(1 - t_{\sigma 1})$. The optimal interior solution is $p_s^* = \frac{\alpha-(\rho-\theta)q}{2}$. The conditions for $0 < t_{\sigma 1} < 1$ and $t_{\sigma 2} \geq 2$ are $2\alpha - (2\rho - \theta - 1)q \leq p_s < \alpha - (\rho - \theta)q$. Checking this condition we see the interior solution holds if $\alpha \leq \frac{(3\rho-\theta-2)q}{3} < (\rho - 1)q$.

Given that OG users choose (Upgrade1+SaaS, Upgrade2), in order for NG users to prefer (New1+SaaS, Upgrade2) rather than (SaaS, New2), we need $(\rho q + 2k)t_{\sigma 1} - p_u \geq \int_0^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)dt$, which is the same condition as (H14). In order for NG users to prefer (New1+SaaS, Upgrade2) rather than (SaaS, SaaS), we need $(\rho q + 2k)t_{\sigma 1} - p_n + (2\rho - 1)q + 2k - p_u \geq \int_0^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)dt + \int_1^2 (\theta q + \alpha t + k - p_s)dt$; that is, $p_u + p_n \leq (2\rho - \theta - 1)q + k + p_s - \frac{3\alpha}{2} + \frac{[p_s+(\rho-\theta)q+2k][p_s+(\rho-\theta)q]}{2\alpha}$ (H17). Given that NG users choose (New1+SaaS, Upgrade2), in order for the OG users to prefer (Upgrade1+SaaS, Upgrade2) rather than (Old+SaaS, Upgrade2), we need $(\rho q + 2k)t_{\sigma 1} - p_u \geq (q + k)t_{\sigma 3} + \int_{t_{\sigma 3}}^{t_{\sigma 1}} (\theta q + \alpha t + k - p_s)\mathrm{d}t$, which is the same condition as (H16).

When $\alpha \leq \frac{(3\rho-\theta-2)q}{3} < (\rho - 1)q$, (H14) binds and we have $p_u = \frac{[(\rho-\theta)q+\alpha+4k][(\rho-\theta)q+\alpha]}{8\alpha}$ and $p_n = \frac{(3\rho-\theta-2)q}{2} - \alpha + k$. Furthermore, $p_u < p_n$.

Now consider the boundary solution. If $\frac{(3\rho-\theta-2)q}{3} < \alpha \leq (\rho - 1)q$, substituting $p_s^* = 2\alpha - (2\rho - \theta - 1)q$ into (H14) we have $p_u = \frac{[2\alpha-(\rho-1)q+2k][2\alpha-(\rho-1)q]}{2\alpha}$, and by (H17), $p_n = k + \frac{\alpha}{2}$.

Case (3) Consider SP3$'''$. Knowing it only serves in one period, the SaaS vendor's optimization problem becomes $2p_s(2 - t_{\sigma 2})$. The optimal interior solution is $p_s^* = \frac{2\alpha-(2\rho-\theta-1)q}{2}$. The conditions for $t_{\sigma 1} \geq 1$ and $1 < t_{\sigma 2} < 2$ are $\alpha - (\rho - \theta)q \leq p_s < 2\alpha - (2\rho - \theta - 1)q$ (H18). Checking this condition we can verify that the interior solution does not hold. So the SaaS vendor prices at boundary price $p_s^* = \alpha - (\rho - \theta)q$. Substituting p_s^* into (H14) we have $p_u = \frac{[2\alpha-(\rho-1)q+2k][2\alpha-(\rho-1)q]}{2\alpha}$. By (H15) we have $p_n = k + \frac{\alpha}{2}$.

We see that in the range $(\rho - \theta)q \leq \alpha < (2\rho - \theta - 1)q$, there are two equilibrium strategies: one symmetric (SP3$'$) and one asymmetric (SP3$''$ or SP3$'''$). It is worth noting that if an equilibrium pricing strategy consists of boundary price, then the equilibrium is unstable because the vendor can easily deviate from the boundary pricing strategy by lowering its price a little bit, and then end up with entering the feasible pricing region of the other equilibrium. If an equilibrium pricing strategy consists of interior solution, it emerges as the final stable equilibrium at which both vendors have no incentive to deviate given the other vendor's strategy. Comparing the equilibrium profits under the different regions, we can establish the equilibrium outcome in the two-period model. We summarize and present the results in Proposition 7, where K_2' and K_3' are determined by solving $\pi_{perp}^{SP3''} = \pi_{perp}^{SP2'}$ and $\pi_{perp}^{SP3'} = \pi_{perp}^{SP2'}$ in their respective segments. We omit their lengthy mathematical expressions here. In summary, we obtain the following equilibrium outcome.

Proposition 7 *(Equilibrium Outcome in the Two-Period Model)*

(a) (Entry Deterrence Equilibrium) If $\alpha \leq (\rho - \theta)q$ and $k > K_1'$, the perpetual software vendor deters the SaaS vendor's entry in both periods. The equilibrium user strategy is [(Upgrade1, Upgrade2), (New1, Upgrade2)]. The perpetual software vendor's equilibrium prices are $p_u^* = (\rho - \theta)q + k - \frac{\alpha}{2}$ and $p_n^* = (2\rho - \theta - 1)q + k - \frac{3\alpha}{2}$.

(b) (Market Segmentation Equilibrium) If i)$\alpha \leq (\rho - \theta)q$ and $k \leq K_1'$, or ii) $(\rho - \theta)q < \alpha \leq \frac{(3\rho-\theta-2)q}{3}$ and $k \leq K_2'$, or iii) $\frac{(3\rho-\theta-2)q}{3} < \alpha < (2\rho - \theta - 1)q$, and $k \leq K_3'$, the perpetual software vendor and the SaaS vendor segment the market. The equilibrium user strategy is [(Upgrade1, Upgrade2), (SaaS, SaaS)], and the equilibrium prices are as follows:

If $\alpha \leq \frac{3(\rho-1)q}{2}$, then $p_u^* = (\rho - 1)q$, $p_n^* = max[\frac{3(\rho-1)q}{2} + 2k - \frac{3\alpha}{2}, 2(\rho - 1)q + 4k]$, and $p_s^* = \frac{(2\theta-\rho-1)q}{2} + k + \alpha$.

If $\alpha > \frac{3(\rho-1)q}{2}$, then $p_u^* = (\rho - 1)q$, $p_n^* = 4k + 2\alpha - 2(\rho - 1)q$, and $p_s^* = 2\alpha + k - (2\rho - \theta - 1)q$.

(c) (Sequential Dominance Equilibrium) i) If $(\rho - \theta)q < \alpha \leq \frac{(3\rho-\theta-2)q}{3}$ and $k > K_2'$, the perpetual software vendor and the SaaS vendor sequentially serve the market. The equilibrium user strategy is [(Upgrade1+SaaS, Upgrade2), (New1+SaaS, Upgrade2)]. The equilibrium prices are: $p_u^* = \frac{[(\rho-\theta)q+\alpha+4k][(\rho-\theta)q+\alpha]}{8\alpha}$, $p_n^* = \frac{(3\rho-\theta-2)q}{2} - \alpha + k$, and $p_s^* = \frac{\alpha-(\rho-\theta)q}{2}$.

ii) If $\frac{(3\rho-\theta-2)q}{3} < \alpha < (2\rho-\theta-1)q$ *and* $k > K_3'$*, the perpetual software vendor and the SaaS vendor sequentially serve the market. The equilibrium user strategy is [(Upgrade1+SaaS, Upgrade2+SaaS), (New1+SaaS, Upgrade2+SaaS)]. The equilibrium prices are as follows:*

If $\alpha \leq (\rho-1)q$*, then* $p_u^* = \frac{[(\rho-2\theta+1)q+3\alpha+8k][(\rho-2\theta+1)q+3\alpha]}{32\alpha}$, $p_n^* = \frac{[(5\rho-2\theta-3)q-\alpha+8k][(5\rho-2\theta-3)q-\alpha]}{32\alpha}$*, and* $p_s^* = \frac{3\alpha-(3\rho-2\theta-1)q}{4}$.

If $\alpha > (\rho-1)q$*, then* $p_u^* = p_n^* = \frac{5\alpha^2+8\alpha k+(24k\rho-16k\theta-2\alpha\rho-4\alpha\theta+6\alpha-8k)q+(13\rho^2-12\rho\theta+4\theta^2-14\rho+4\theta+5)q^2}{32\alpha}$*, and* $p_s^* = \frac{3\alpha-(3\rho-2\theta-1)q}{4}$.

Appendix I

SaaS Vendor's Quality Improvement Cost

Proposition 8 *(Entry Deterrence Equilibrium with* c_α*) The perpetual software vendor deters the SaaS vendor's entry when the network effect is strong enough or when the SaaS quality improvement cost is high enough. The equilibrium user strategy is SP1 (Upgrade, New), where the OG users upgrade and the NG users adopt the new perpetual software. The equilibrium prices are as follows:*

(a) If $c_\alpha \leq \frac{\alpha}{2} + (\theta-1)q$ *and* $k \geq K_1' = \frac{\alpha-(\rho-2\theta+1)q-2c_\alpha}{2}$*, then* $p_u^{c_\alpha} = p_n^{c_\alpha} = (\rho-\theta)q + k - \frac{\alpha}{2} + c_\alpha$.

(b) If $c_\alpha > \frac{\alpha}{2} + (\theta-1)q$*, then* $p_u^{c_\alpha} = (\rho-1)q + k$ *and* $p_n^{c_\alpha} = (\rho-\theta)q + k - \frac{\alpha}{2} + c_\alpha$.

Proof. Consider SP1 (Upgrade, New). Similar to the Proof of Proposition 2, we must ensure that the OG users prefer upgrading to the new version rather than continuing to use the old version, which requires $\rho q + 2k - p_u \geq q + k$; that is, $p_u \leq (\rho-1)q + k$ (I1). Meanwhile, the perpetual software vendor needs to make sure that OG users prefer upgrading rather than adopting SaaS, even if the SaaS price is reduced to the lowest level $p_s = c_\alpha$. That is, the entry deterrence condition is $\rho q + 2k - p_u \geq \int_0^1 (\theta q + \alpha t + k - c_\alpha)dt$, so that $p_u \leq (\rho-\theta)q + k - \frac{\alpha}{2} + c_\alpha$ (I2). Similarly, to ensure that NG users prefer the new perpetual software to the SaaS at $p_s = c_\alpha$, the condition is $\rho q + 2k - p_n \geq \int_0^1 (\theta q + \alpha t + k - c_\alpha)dt$; that is, $p_n \leq (\rho-\theta)q + k - \frac{\alpha}{2} + c_\alpha$ (I3).

If $c_\alpha \leq \frac{\alpha}{2} + (\theta-1)q$, (I2) is binding. Because $p_u \leq p_n$, by (I2) and (I3) the perpetual software vendor sets the prices at respective upper bounds: $p_n^{SP1} = p_u^{SP1} = (\rho-\theta)q + k - \frac{\alpha}{2} + c_\alpha$. Consequently, we get the perpetual software vendor's profit $\pi_{perp}^{SP1} = 2(\rho-\theta)q + 2k - \alpha + 2c_\alpha$.

If $c_\alpha > \frac{\alpha}{2} + (\theta-1)q$, (I1) is binding. By (I2) and (I3) we have $p_u^{SP1} = (\rho-1)q + k$ and $p_n^{SP1} = (\rho-\theta)q + k - \frac{\alpha}{2} + c_\alpha$. Consequently, we get the perpetual software vendor's profit $\pi_{perp}^{SP1} = (2\rho-\theta-1)q + 2k - \frac{\alpha}{2} + c_\alpha$.

Consider SP2 (Upgrade, SaaS). Similar to the Proof of Proposition 3, we have $p_u \leq (\rho-1)q$ (I4); $p_u \leq p_s + (\rho-\theta)q - k - \frac{\alpha}{2}$ (I5); and $p_n \geq p_s + (\rho-\theta)q + k - \frac{\alpha}{2}$ (I6).

To maximize its profit, the perpetual software vendor sets p_n as high as possible so that the SaaS vendor can also charge a high enough price p_s, which in turn allows the perpetual software vendor to charge a high upgrade price p_u. As a result, the perpetual software vendor charges $p_u = (\rho-1)q$ to make the OG users' IC constraint (I4) binding. If $\alpha \leq 2(\rho-1)q$, the SaaS vendor charges as much as $p_s^{SP2} = (\theta-1)q + k + \frac{\alpha}{2}$ by (I5), and by (I6) $p_n^{SP2} = (\rho-1)q + 2k$. If $\alpha > 2(\rho-1)q$, then the boundary solution $p_s^{SP2} = \alpha + k - (\rho-\theta)q$ as specified in Table C1 holds. By (I4) and (I5) $p_u^{SP2} = (\rho-1)q$ and by (I6) $p_n^{SP2} = \frac{\alpha}{2} + k$. So $\pi_{perp}^{SP2} = (\rho-1)q$.

Finally, we compare the perpetual software vendor's profits under SP1 and SP2. We can show that, if $c_\alpha \leq \frac{\alpha}{2} + (\theta-1)q$, then $\pi_{perp}^{SP1} > \pi_{perp}^{SP2}$ if $k > \frac{\alpha-(\rho-2\theta+1)q-2c_\alpha}{2}$. If $c_\alpha > \frac{\alpha}{2} + (\theta-1)q$, then $\pi_{perp}^{SP1} > \pi_{perp}^{SP2}$.

Appendix J

OG User's Switching Cost

Proposition 9 *(Equilibria with OG User Switching Cost) Both the SaaS quality improvement rate α and users' switching cost c affect the equilibrium outcome as follows:*

(a) (Entry Deterrence Equilibrium) If $\alpha \leq A_1$, the perpetual software vendor deters the SaaS vendor's entry. The equilibrium user strategy is SP1 (Upgrade, New). The perpetual software vendor's equilibrium prices are $p_u^* = p_n^* = (\rho - \theta)q - \frac{\alpha}{2}$.

(b) (Market Segmentation Equilibrium) The perpetual software vendor and the SaaS vendor segment the market. The equilibrium user strategy is SP2 (Upgrade, SaaS).

If i) $A_1 < \alpha \leq A_2$, or ii) $\alpha > A_2$ and $C_2 < c \leq C_1$, then equilibrium prices are $p_u^* = p_n^* = (\rho - 1)q$ and $p_s^* = (\theta - 1)q + \frac{\alpha}{2}$.

If $\alpha \leq A_3$ and $c \leq C_2$, then equilibrium prices are $p_u^* = (\rho - 1)q$, $p_n^* = \frac{\alpha}{2} - \sqrt{2\alpha c}$, and $p_s^* = \alpha - (\rho - \theta)q - \sqrt{2\alpha c}$.

(c) (Competitive Lock-in Equilibrium) If $\alpha > A_2$ and $c > C_1$, the perpetual software vendor serves the OG users over the whole time interval $[0,1]$ and NG users in the time interval $[0, \frac{\alpha+(\rho-\theta)q}{2\alpha}]$. The SaaS vendor serves the NG users in the time interval $[\frac{\alpha+(\rho-\theta)q}{2\alpha}, 1]$. The equilibrium user strategy is SP7 (Upgrade, New+SaaS). The equilibrium prices are $p_u^* = p_n^* = \frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$ and $p_s^* = \frac{\alpha-(\rho-\theta)q}{2}$.

(d) (Sequential Dominance Equilibrium) If $\alpha > A_3$ and $c \leq C_2$, the perpetual software vendor serves both OG and NG users in the time interval $[0, \frac{\alpha+(\rho-\theta)q}{2\alpha}]$, and the SaaS vendor serves both OG and NG users in the time interval $[\frac{\alpha+(\rho-\theta)q}{2\alpha}, 1]$. The equilibrium user strategy is SP6 (Upgrade+SaaS, New+SaaS). The equilibrium prices are $p_u^* = \frac{(\rho-1)q[\alpha-(\theta-1)q]}{2\alpha}$, $p_n^* = \frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$, and $p_s^* = \frac{\alpha-(\rho-\theta)q}{2}$.

Our proof involves several steps. First, given user strategies, we analyze four sub-game perfect equilibria and the corresponding vendor prices and profits. Then we derive the final equilibrium outcome under different market conditions.

Entry Deterrence Strategy

Note that SP1 (Upgrade, New) can only occur when $\alpha \leq (\rho - \theta)q$. That is, the quality of SaaS does not exceed the quality of the new perpetual software at the end of the product life cycle.

Given that NG users purchase the new perpetual software, OG users prefer to upgrade rather than continue to use the old version. So we have $p_u \leq (\rho - 1)q$ (J1). Also, OG users prefer to upgrade rather than opt for SaaS. Note that moving to SaaS incurs additional switching costs c. So we get $p_u \leq (\rho - \theta)q - \frac{\alpha}{2} + c$ (J2).

Given that OG users upgrade, NG users prefer to buy the new perpetual software rather than SaaS. This situation gives us $p_n \leq (\rho - \theta)q - \frac{\alpha}{2}$ (J3). In addition, we have the constraint $p_n \geq p_u$.

Putting all these constraints together, we get the perpetual software vendir's prices $p_u^{SP1} = p_n^{SP1} = (\rho - \theta)q - \frac{\alpha}{2}$ and profit $\pi_{perp}^{SP1} = 2(\rho - \theta)q - \alpha$.

Market Segmentation Strategy

Consider SP2 (Upgrade, SaaS), where the perpetual software vendor allows the SaaS vendor to enter the market. It can happen under both $\alpha \leq (\rho - \theta)q$ and $\alpha > (\rho - \theta)q$.

Case (1) $\alpha \leq (\rho - \theta)q$. Given that NG users choose SaaS, we need to ensure that, for OG users, upgrading is better than using the old version and also better than SaaS. Thus, (J1) and $p_s \geq p_u - (\rho - \theta)q + \frac{\alpha}{2} - c$ (J4) must hold. Similarly, NG users prefer SaaS to the new perpetual software, and so $p_s \le p_n - (\rho - \theta)q + \frac{\alpha}{2}$ (J5). In addition, $p_n \ge p_u$. So we get $p_u^{SP2} = p_n^{SP2} = (\rho - 1)q$, $p_s^{SP2} = (\theta - 1)q + \frac{\alpha}{2}$. Vendor profits are $\pi_{perp}^{SP2} = (\rho - 1)q$ and $\pi_{SaaS}^{SP2} = (\theta - 1)q + \frac{\alpha}{2}$.

Case (2) $\alpha > (\rho - \theta)q$. When α is large, the SaaS becomes competitive, and switching becomes possible. We first derive the non-switching (NS) condition for OG users. Conditional on the fact that OG users switch, the switching time is when the net payoff from SaaS exceeds the the net payoff from the new version of perpetual software. Similar to the baseline case, $t_{s1} = \frac{p_s - (\theta - 1)q}{\alpha}$. Taking into account the switching cost, the condition for switching is $\int_{t_{s1}}^{1} (\theta q + \alpha t - p_s)dt - \rho q(1 - t_{s1}) \ge c$. Substituting into t_{s1} and solving this inequality, we get $p_s \ge \alpha - (\rho - \theta)q - \sqrt{2\alpha c}$ (NS).

We can verify that the SaaS price derived in Case (1) satisfies this (NS) condition when $\sqrt{2\alpha c} \ge \frac{\alpha}{2} - (\rho - 1)q$; that is, $c \ge \frac{[\alpha - 2(\rho - 1)q]^2}{8\alpha} \doteq C_2$. Therefore the same optimal solutions apply.

When $c < C_2$, however, the (NS) condition is binding, so $p_s = \alpha - (\rho - \theta)q - \sqrt{2\alpha c}$. Reexamining the incentive compatibility conditions (J1), (J4), and (J5), we get $p_u^{SP2} = (\rho - 1)q$, $p_n^{SP2} = \frac{\alpha}{2} - \sqrt{2\alpha c}$ by (J3), and $p_u^{SP2} < p_n^{SP2}$. The vendor's profits are $\pi_{perp}^{SP2} = (\rho - 1)q$ and $\pi_{SaaS}^{SP2} = \alpha - (\rho - \theta)q - \sqrt{2\alpha c}$.

Competitive Lock-In Strategy

Consider a new strategy pair (Upgrade, New+SaaS). We denote it as SP7. It occurs under the condition $\alpha > (\rho - \theta)q$, where the SaaS quality outperforms the perpetual software quality at some time $t \in [0,1]$. To ensure that OG users do not switch, the (NS) condition must hold. And to ensure that NG users switch, the net payoff from SaaS must be higher than the net payoff from the new perpetual software by time $t = 1$; that is, $\theta q + \alpha - p_s \ge \rho q$. So $p_s \le \alpha - (\rho - \theta)q$ (J6). In addition, NG users switch at $t_{s3} = \frac{(\rho - \theta)q + p_s}{\alpha}$. The SaaS vendor's profit thus is expressed as $p_s(1 - t_{s3})$, and the optimal SaaS price is $p_s^* = \frac{\alpha - (\rho - \theta)q}{2}$. Accordingly, the optimal switching time is $t_{s3}^* = \frac{\alpha + (\rho - \theta)q}{2\alpha}$. There are two cases:

Case (1) When $c \ge \frac{[\alpha - (\rho - \theta)q]^2}{8\alpha}$, the interior solution $p_s^{SP7} = \frac{\alpha - (\rho - \theta)q}{2}$ satisfies both (NS) and (J6). We now check the incentive compatibility conditions for both groups of users. Given that OG users upgrade, NG users prefer New+SaaS over SaaS if $\rho q t_{s1}^* - p_n + \int_{t_{s1}^*}^{1} (\theta q + \alpha t - p_s^*)dt \ge \int_0^1 (\theta q + \alpha t - p_s^*)dt$ (J7). Substituting into p_s^{SP7} and t_{s3}^* and simplifying the condition, we get $p_n \le \frac{[\alpha + (\rho - \theta)q]^2}{8\alpha} \doteq p_1$. Similarly, given that NG users choose New+SaaS, OG users prefer Upgrade over Old+SaaS if $\rho q - p_u \ge q t_{s1}^* + \int_{t_{s1}^*}^{1} (\theta q + \alpha t - p_s^*)dt - c$ (J8), where t_{s1}^* is the switching time if OG users switch from the old version of perpetual software to SaaS, and t_{s1}^* is given by $\theta q + \alpha t_{s1}^* - p_s^* = q$. Using p_s^{SP7}, we have $t_{s1}^* = \frac{\alpha - (\rho + \theta - 2)q}{2\alpha}$. If $\alpha < (\rho + \theta - 2)q$, (J8) is satisfied. So $p_n = p_u = \frac{[\alpha + (\rho - \theta)q]^2}{8\alpha}$. If $\alpha \ge (\rho + \theta - 2)q$, $t_{s1}^* > 0$. Substituting t_{s1}^* into (J8) we get $p_u \le (\rho - 1) - \frac{[\alpha + (\rho + \theta - 2)q]^2}{8\alpha} + c \doteq p_2$. When $c = \frac{[\alpha - (\rho - \theta)q]^2}{8\alpha}$, $p_2 < p_1$. Because p_2 linearly increases in c, there is a threshold value $c^* = \frac{[\alpha + (\rho - \theta)q]^2 + [\alpha + (\rho + \theta - 2)q]^2}{8\alpha} - (\rho - 1)q$ such that, for $\frac{[\alpha - (\rho - \theta)q]^2}{8\alpha} \le c < c^*$, $p_1 > p_2$; thus, $p_n^{SP7} = \frac{[\alpha + (\rho - \theta)q]^2}{8\alpha}$ and $p_u^{SP7} = (\rho - 1)q - \frac{[\alpha + (\rho + \theta - 2)q]^2}{8\alpha} + c$; and for $c \ge c^*$, $p_1 < p_2$; thus, $p_n^{SP7} = p_u^{SP7} = \frac{[\alpha + (\rho - \theta)q]^2}{8\alpha}$.

Case (2) When $c < \frac{[\alpha - (\rho - \theta)q]^2}{8\alpha}$, we have a boundary solution $p_s^{SP7} = \alpha - (\rho - \theta)q - \sqrt{2\alpha c}$; accordingly, the switching time becomes $t_{s3}^* = \frac{\alpha - \sqrt{2\alpha c}}{\alpha}$. We next check users' incentive compatibility conditions. Condition (J7) becomes $p_n \le \frac{\alpha}{2} - \sqrt{2\alpha c} + c \doteq p_1$. For condition (J8), $t_{s1}^* = \frac{p_s^* - (\theta - 1)q}{\alpha} = \frac{\alpha - (\rho - 1)q - \sqrt{2\alpha c}}{\alpha}$. If $\alpha < (\rho - 1)q$, or if $\alpha \ge (\rho - 1)q$ and $c > \frac{[\alpha - (\rho - 1)q]^2}{2\alpha}$, (J8) is satisfied. In these cases, $p_n^{SP7} = p_u^{SP7} = \frac{\alpha}{2} - \sqrt{2\alpha c} + c$. If $\alpha \ge (\rho - 1)q$ and $c < \frac{[\alpha - (\rho - 1)q]^2}{2\alpha}$, $t_{s1}^* > 0$, substituting t_{s1}^* into (J8), we get $p_u \le (\rho - 1) - \frac{[(\rho - 1)q + \sqrt{2\alpha c}]^2}{2\alpha} + c \doteq p_2$. We can verify that $p_1 > p_2$. Hence, $p_n^{SP7} = \frac{\alpha}{2} - \sqrt{2\alpha c} + c$, $p_u^{SP7} = (\rho - 1) - \frac{[(\rho - 1)q + \sqrt{2\alpha c}]^2}{2\alpha} + c$, and $p_n^{SP7} > p_u^{SP7}$.

Sequential Dominance Strategy

This strategy pair is SP6 (Upgrade+SaaS, New+SaaS). It occurs under the condition $\alpha > (\rho - \theta)q$. To ensure that OG users switch to SaaS, the switching condition is $p_s < \alpha - (\rho - \theta)q - \sqrt{2\alpha c}$ (J9), and note that when this condition holds, NG users also switch. Similar to the baseline model, the switching time is $t_{s3} = \frac{(\rho-\theta)q+p_s}{\alpha}$. The SaaS vendor's profit thus is expressed as $2p_s(1-t_{s3})$, and the optimal SaaS price is $p_s^* = \frac{\alpha-(\rho-\theta)q}{2}$. Accordingly, the optimal switching time is $t_{s3}^* = \frac{\alpha+(\rho-\theta)q}{2\alpha}$. We get three cases:

Case (1) When $c < \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$, the internal optimal solution $p_s^{SP6} = \frac{\alpha-(\rho-\theta)q}{2}$ satisfies (J9). The solution is the same as the baseline model, as in Proposition 4.

Case (2) When $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} \le c < \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$, we derive the boundary solution $p_s^{SP6} = \alpha-(\rho-\theta)q-\sqrt{2\alpha c}$; accordingly, the switching time becomes $t_{s3}^* = \frac{\alpha-\sqrt{2\alpha c}}{\alpha}$. We reexamine the incentive compatibility conditions. Given that OG users choose Upgrade+SaaS, NG users prefer New+SaaS over SaaS if $\rho q t_{s3}^* - p_n + \int_{t_{s3}^*}^1(\theta q+\alpha t-p_s^*)dt \ge \int_0^1(\theta q+\alpha t-p_s^*)dt$. So we get $p_n \le \frac{[\alpha-\sqrt{2\alpha c}]^2}{2\alpha} \doteq p_1$. Given that NG users choose New+SaaS, OG users prefer Upgrade+SaaS over Old+SaaS if $\rho q t_{s2}^* - p_u + \int_{t_{s2}^*}^1(\theta q+\alpha t-p_s^*)dt - c \ge q t_{s1}^* + \int_{t_{s1}^*}^1(\theta q+\alpha t-p_s^*)dt - c$ (J10), where $t_{s1}^* = \frac{p_s^{SP6}-(\theta-1)q}{\alpha} = \frac{\alpha-(\rho-1)q-\sqrt{2\alpha c}}{\alpha}$. We note that $t_{s1}^* < 0$ when $\alpha < (\rho+\theta-2)q$, or $\alpha > (\rho+\theta-2)q$ and $\frac{[\alpha-(\rho-1)q]^2}{2\alpha} \le c < \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$. So (J10) is satisfied and $p_n^{SP6} = p_u^{SP6} = \frac{[\alpha-\sqrt{2\alpha c}]^2}{2\alpha}$. When $\alpha \ge (\rho+\theta-2)q$ and $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} \le c < \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$, $t_{s1}^* > 0$. Substituting t_{s1}^* into (J10), we get $p_u \le \frac{(\rho-1)q(\alpha-\sqrt{2\alpha c})}{\alpha} - \frac{[(\rho-1)q]^2}{2\alpha} \doteq p_2$. Because $p_1 - p_2 = \frac{[\alpha-(\rho-1)q-\sqrt{2\alpha c}]^2}{2\alpha} > 0$, we have $p_n^{SP6} = \frac{[\alpha-\sqrt{2\alpha c}]^2}{2\alpha}$, $p_u^{SP6} = \frac{(\rho-1)q(\alpha-\sqrt{2\alpha c})}{\alpha} - \frac{[(\rho-1)q]^2}{2\alpha}$, and $p_n^{SP6} > p_u^{SP6}$.

Case (3) When $c \ge \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$, the condition (J9) does not hold. Thus, SP6 does not appear.

Profit Comparison in All Parameter Regions

To see which strategy pair is the equilibrium, we need to compare the vendor's profits. When $\alpha \le (\rho-\theta)q$, both SP1 and SP2 are possible; when $\alpha > (\rho-\theta)q$, SP2, SP6, and SP7 are possible. Using Table 3, we have in total 10 parameter regions to study. In the following, we examine one region to show how we obtain the equilibrium; for all the rest of the comparisons, the analysis is similar.

Consider the parameter region $\alpha \ge (\rho-\theta)q$, $\max\left\{\frac{[\alpha-(\rho-1)q]^2}{2\alpha}, \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}\right\} < c < \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$. In this region, SP2, SP6, and SP7 are all feasible strategies. Vendor profits are $\pi_{perp}^{SP2} = (\rho-1)q$, $\pi_{SaaS}^{SP2} = (\theta-1)q + \frac{\alpha}{2}$ in SP2, $\pi_{perp}^{SP6} = \frac{[\alpha-\sqrt{2\alpha c}]^2}{\alpha}$, $\pi_{SaaS}^{SP6} = \frac{2[\alpha-(\rho-\theta)q-\sqrt{2\alpha c}]\sqrt{2\alpha c}}{\alpha}$ in SP6, and $\pi_{perp}^{SP7} = \frac{[\alpha+(\rho-\theta)q]^2}{4\alpha}$, $\pi_{SaaS}^{SP7} = \frac{[\alpha-(\rho-\theta)q]^2}{4\alpha}$ in SP7, respectively.

We first compare SP6 and SP7. Because $\Delta\pi_{perp}^{SP7-6} = \frac{[3\alpha+(\rho-\theta)q-2\sqrt{2\alpha c}][-\alpha+(\rho-\theta)q+2\sqrt{2\alpha c}]}{4\alpha} > 0$, the perpetual software vendor prefers SP7 to SP6. For the SaaS vendor, we find that $\partial\Delta\pi_{SaaS}^{SP7-6}/\partial c = \frac{2[(\rho-\theta)q+2\sqrt{2\alpha c}-\alpha]}{\sqrt{2\alpha c}} > 0$. If $(\rho-\theta)q < \alpha < (\rho+\theta-2)q$, $\underline{c} = \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$ and $\overline{c} = \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$. If $\alpha \ge (\rho+\theta-2)q$, $\underline{c} = \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$ and $\overline{c} = \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$. We can show that $\Delta\pi_{SaaS}^{SP7-6} < 0$ at $\underline{c}$ and $\Delta\pi_{SaaS}^{SP7-6} > 0$ at $\overline{c}$. So a value C_1 must exist in this parameter region such that $\Delta\pi_{SaaS}^{SP7-6} = 0$ at C_1. Solving the equation, we get $C_1 = \frac{(\sqrt{2}+1)^2[\alpha-(\rho-\theta)q]^2}{16\alpha}$. For $c < C_1$, $\Delta\pi_{SaaS}^{SP7-6} < 0$, meaning that the SaaS vendor prefers SP6 to SP7 and so reduces its price to deviate to SP6. Meanwhile, for $c > C_1$, $\Delta\pi_{SaaS}^{SP7-6} > 0$ meaning that the SaaS vendor prefers SP7 to SP6.

We next compare SP2 with SP6 when $c < C_1$, and we compare SP2 with SP7 when $c > C_1$.

Case (1) $c < C_1$. For the SaaS vendor, $\partial\Delta\pi_{SaaS}^{SP6-2}/\partial c = \frac{-2[(\rho-\theta)q+2\sqrt{2\alpha c}-\alpha]}{\sqrt{2\alpha c}} < 0$; and $\Delta\pi_{SaaS}^{SP6-2} < 0$ at $c = \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$. Because in this region all $c \ge \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$, we conclude that $\Delta\pi_{SaaS}^{SP6-2} < 0$ in the whole region. Thus, the SaaS vendor always prefers SP2. For the perpetual software vendor, $\Delta\pi_{perp}^{SP6-2} = \alpha + 2c - 2\sqrt{2\alpha c} - (\rho-1)q$. We solve $\Delta\pi_{perp}^{SP6-2} = 0$ and get two solutions: $c_1 = \frac{[\sqrt{\alpha}-\sqrt{(\rho-1)q}]^2}{2}$ and $c_2 = \frac{[\sqrt{\alpha}+\sqrt{(\rho-1)q}]^2}{2}$. We can further prove that $c_1 < \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$ and $c_2 > C_1$, and so both roots are outside this region. Hence, $\Delta\pi_{perp}^{SP6-2} < 0$, meaning that the perpetual software vendor prefers SP2. We conclude that when $c < C_1$, the final equilibrium is SP2.

Case (2) $c > C_1$. For the SaaS vendor, $\partial\Delta\pi_{SaaS}^{SP7-2}/\partial\alpha = \frac{-4[\alpha^2+(\rho-\theta)^2q^2]}{16\alpha^2} < 0$; and $\Delta\pi_{SaaS}^{SP7-2} < 0$ at $\alpha = (\rho-\theta)q$. Because in this region all $\alpha \ge (\rho-\theta)q$, we conclude that $\Delta\pi_{SaaS}^{SP7-2} < 0$ in the whole region. Thus, the SaaS vendor always prefers SP2. For the perpetual software vendor, $\Delta\pi_{perp}^{SP7-2} = \frac{[\alpha+(\rho-\theta)q]^2}{4\alpha} - (\rho-1)q$. We can show that $\Delta\pi_{perp}^{SP7-2} = (1-\theta)q < 0$ at $\alpha = (\rho-\theta)q$, which is the smallest α in this region, and that $\partial\Delta\pi_{perp}^{SP7-2}/\partial\alpha = \frac{4[\alpha+(\rho-\theta)q][\alpha-(\rho-\theta)q]}{16\alpha^2} > 0$. Solving $\Delta\pi_{perp}^{SP7-2} = 0$ for α, we get two solutions: $A_1 = (\rho+\theta-2)q - 2\sqrt{(\rho-1)(\theta-1)}q$ and $A_2 = (\rho+\theta-2)q + 2\sqrt{(\rho-1)(\theta-1)}q$. Note that $A_1 < (\rho-\theta)q$, so it falls outside of the region, and $(\rho+\theta-2)q < A_2 < 2(\rho-1)q$, so it falls within the region. Therefore, when $\alpha < A_2$, $\Delta\pi_{perp}^{SP7-2} < 0$, meaning that the perpetual software vendor prefers SP2 and that, in this sub-region, SP2 is the equilibrium outcome. When $\alpha \geq A_2$, $\Delta\pi_{perp}^{SP7-2} > 0$, meaning that the perpetual software vendor prefers SP7. The perpetual software vendor thus reduces prices to deviate from SP2 to SP7. We conclude that in this sub-region, SP7 is the equilibrium outcome.

Finally, after combining all the conditions and equilibrium results, we obtain the four equilibria shown in Proposition 9 and Table J1.

Table J1. Parameter Conditions, Prices, and Profits Under Switching Cost Model

(a) Parameter Conditions with Switching Costs

Strategy Pairs	Regions	Parameter Conditions
SP1	1	$\alpha < (\rho-\theta)q$
SP2	2	(1) $\alpha < (\rho-\theta)q$; (2) $\alpha \geq (\rho-\theta)q$, $c \geq C_2 = \frac{[\alpha-2(\rho-1)q]^2}{8\alpha}$
	3	$\alpha \geq (\rho-\theta)q$, $c < C_2 = \frac{[\alpha-2(\rho-1)q]^2}{8\alpha}$
SP6	4	$(\rho-\theta)q \leq \alpha < (\rho+\theta-2)q$, $c < \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$
	5	$\alpha \geq (\rho+\theta-2)q$, $c < \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$
	6	(1) $(\rho-\theta)q \leq \alpha < (\rho+\theta-2)q$; (2) $\alpha \geq (\rho+\theta-2)q$, $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} \leq c < \frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$
	7	$\alpha > (\rho+\theta-2)q$, $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} < c < \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$
SP7	8	(1) $(\rho-\theta)q \leq \alpha < (\rho+\theta-2)q$, $c \geq \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$; (2) $\alpha \geq (\rho+\theta-2)q$, $c \geq c^*$
	9	$\alpha > (\rho+\theta-2)q$, $\frac{[\alpha-(\rho-\theta)q]^2}{8\alpha} \leq c < c^* = \frac{[\alpha+(\rho-\theta)q]^2+[\alpha+(\rho+\theta-2)q]^2}{8\alpha} - (\rho-1)q$
	10	(1) $(\rho-\theta)q \leq \alpha < (\rho-1)q$; (2) $(\rho-\theta)q \leq \alpha < (\rho+\theta-2)q$, $\frac{[\alpha-(\rho-1)q]^2}{2\alpha} \leq c < \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$
	11	(1) $(\rho-1)q \leq \alpha < (\rho+\theta-2)q$, $c < \frac{[\alpha-(\rho-1)q]^2}{2\alpha}$; (2) $\alpha \geq (\rho+\theta-2)q$, $c < \frac{[\alpha-(\rho-\theta)q]^2}{8\alpha}$

(b) Optimal Prices with Switching Costs

Strategy Pairs	Regions	p_u	p_n	p_s
SP1	1	$(\rho-\theta)q - \frac{\alpha}{2}$	$(\rho-\theta)q - \frac{\alpha}{2}$	—
SP2	2	$(\rho-1)q$	$(\rho-1)q$	$(\theta-1)q + \frac{\alpha}{2}$
	3	$(\rho-1)q$	$\frac{\alpha}{2} - \sqrt{2\alpha c}$	$\alpha - (\rho-\theta)q - \sqrt{2\alpha c}$
SP6	4	$\frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$	$\frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$	$\frac{\alpha-(\rho-\theta)q}{2}$
	5	$\frac{(\rho-1)q[\alpha-(\theta-1)q]}{2\alpha}$	$\frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$	$\frac{\alpha-(\rho-\theta)q}{2}$
	6	$\frac{[\alpha-\sqrt{2\alpha c}]^2}{2\alpha}$	$\frac{[\alpha-\sqrt{2\alpha c}]^2}{2\alpha}$	$\alpha - (\rho-\theta)q - \sqrt{2\alpha c}$
	7	$\frac{(\rho-1)q(\alpha-\sqrt{2\alpha c})}{\alpha} - \frac{[(\rho-1)q]^2}{2\alpha}$	$\frac{[\alpha-\sqrt{2\alpha c}]^2}{2\alpha}$	$\alpha - (\rho-\theta)q - \sqrt{2\alpha c}$
SP7	8	$\frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$	$\frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$	$\frac{\alpha-(\rho-\theta)q}{2}$
	9	$(\rho-1)q - \frac{[\alpha+(\rho+\theta-2)q]^2}{8\alpha} + c$	$\frac{[\alpha+(\rho-\theta)q]^2}{8\alpha}$	$\frac{\alpha-(\rho-\theta)q}{2}$
	10	$\frac{\alpha}{2} - \sqrt{2\alpha c} + c$	$\frac{\alpha}{2} - \sqrt{2\alpha c} + c$	$\alpha - (\rho-\theta)q - \sqrt{2\alpha c}$
	11	$(\rho-1)q - \frac{[(\rho-1)q+\sqrt{2\alpha c}]^2}{2\alpha} + c$	$\frac{\alpha}{2} - \sqrt{2\alpha c} + c$	$\alpha - (\rho-\theta)q - \sqrt{2\alpha c}$

(c) Optimal Profits with Switching Costs			
Strategy Pairs	**Regions**	π_{perp}	π_{SaaS}
SP1	1	$2(\rho-\theta)q-\alpha$	—
SP2	2	$(\rho-1)q$	$(\theta-1)q+\frac{\alpha}{2}$
	3	$(\rho-1)q$	$\alpha-(\rho-\theta)q-\sqrt{2\alpha c}$
SP6	4	$\frac{[\alpha+(\rho-\theta)q]^2}{4\alpha}$	$\frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$
	5	$\frac{[\alpha+(\rho-\theta)q]^2+4(\rho-1)q[\alpha-(\theta-1)q]}{8\alpha}$	$\frac{[\alpha-(\rho-\theta)q]^2}{2\alpha}$
	6	$\frac{[\alpha-\sqrt{2\alpha c}]^2}{\alpha}$	$\frac{2[\alpha-(\rho-\theta)q-\sqrt{2\alpha c}]\sqrt{2\alpha c}}{\alpha}$
	7	$\frac{[(\alpha-\sqrt{2\alpha c})+2(\rho-1)q](\alpha-\sqrt{2\alpha c})-[(\rho-1)q]^2}{2\alpha}$	$\frac{2[\alpha-(\rho-\theta)q-\sqrt{2\alpha c}]\sqrt{2\alpha c}}{\alpha}$
SP7	8	$\frac{[\alpha+(\rho-\theta)q]^2}{4\alpha}$	$\frac{[\alpha-(\rho-\theta)q]^2}{4\alpha}$
	9	$(\rho-1)q+c+\frac{\alpha+(\rho+\theta-2)q}{4\alpha}$	$\frac{[\alpha-(\rho-\theta)q]^2}{4\alpha}$
	10	$\alpha-2\sqrt{2\alpha c}+2c$	$\frac{[\alpha-(\rho-\theta)q-\sqrt{2\alpha c}]\sqrt{2\alpha c}}{\alpha}$
	11	$(\theta-1)q+\alpha-\frac{[(\rho-1)q+\sqrt{2\alpha c}]^2}{2\alpha}-\sqrt{2\alpha c}+c$	$\frac{[\alpha-(\rho-\theta)q-\sqrt{2\alpha c}]\sqrt{2\alpha c}}{\alpha}$

Appendix K

Continuous NG User Arrival Model

We extend our model to account for NG users' continuous arrival time. We still focus on the vendors' price competition on the planning horizon $[0,1]$. The model setup is the same as the baseline model, except that we assume the NG users with mass 1 uniformly and continuously enter the market on the time interval $[0,1]$. Upon arrival, each NG user makes the software adoption decision for a limited use period, which is normalized to 1. Thus, users who arrive at $t<1$ make a decision based on their expected utility from the software use in the period $[t,1+t]$. We use this model setup for several reasons. First, a decision period of the same length provides a fair comparison among all users. Second, the rapid technological obsolescence makes the software value in the far distant future negligible. To cope with the late arrival users' decision making in the extended time period beyond $t=1$, we assume that the SaaS software quality continues to increase at rate α after time 1. And at $t=1$, the perpetual software vendor releases another "newer" software version with a higher quality. We assume the quality improvement between two major software releases remains the same (i.e., $(\rho-1)q$). Therefore, the "newer" perpetual software's quality can be calculated as $\rho q+(\rho-1)q=(2\rho-1)q$. The continuous user arrival model is depicted in Figure K1.

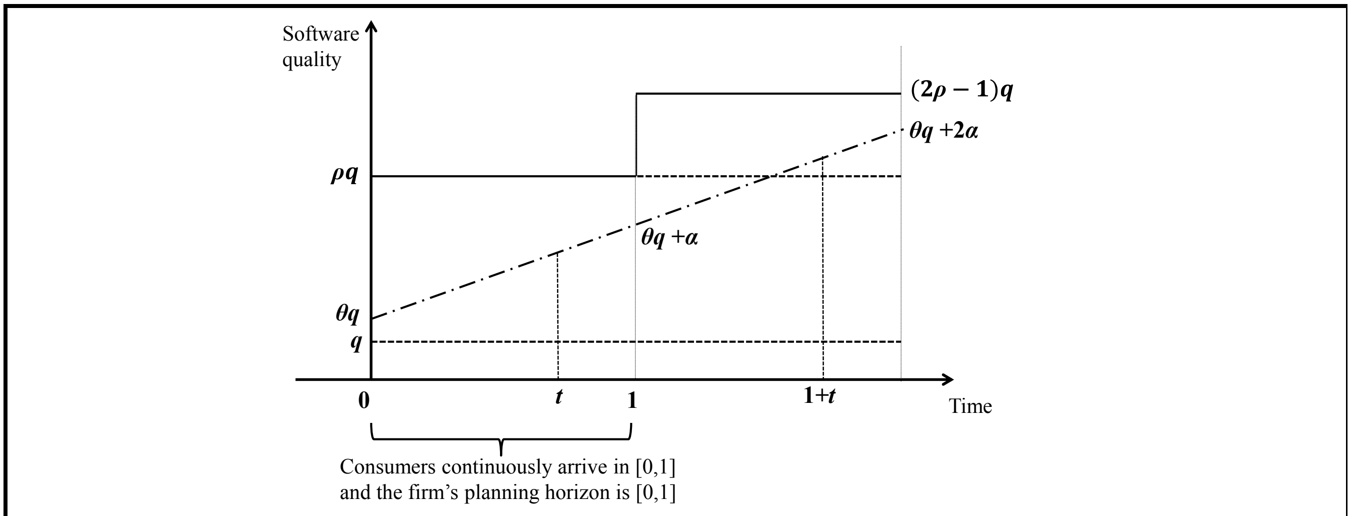

Figure K1. Software Quality Improvement Over Time

In such a dynamic market environment, the installed user base for a software product continues to change. Users who arrive at different times face different expected network values based on both the current number of users and the anticipated future number of users. Even if the current network size is observable, forming the expectation of future network growth is cognitively challenging because it depends on future users' adoption decisions. We therefore omit the network effect in this continuous arrival model (i.e., $k = 0$).

All OG users' strategies are the same as in the baseline model in the "User Utility Definition and Strategy Analysis" section of the paper. For each NG user with arrival time $t < 1$, we note five possible strategies.

New: The user purchases the new perpetual software at price p_n at time t and uses it over the entire period $[t, 1+t]$. The utility is $\rho q - p_n$.

New+Newer: The user purchases the new perpetual software at price p_n at time t, uses it in $[t, 1]$, and then pays an upgrade price p_u to get the newer version at time 1 and uses it for the remaining period $[1, 1+t]$. The utility is $\rho q(1-t) - p_n + (2\rho - 1)qt - p_u$.

New + SaaS: The user purchases the new perpetual software at price p_n at time t and uses it in $[t, t_{s3}]$. It switches to SaaS in the period $[t_{s3}, 1+t]$. The utility is $\rho q(t_{s3} - t) - p_n + \int_{t_{s3}}^{1+t}(\theta q + \alpha t - p_s)dt$.

SaaS: The user uses the SaaS software over the entire period $[t, 1+t]$. The utility is $\int_t^{1+t}(\theta q + \alpha t - p_s)dt$.

SaaS+Newer: The user uses the SaaS software in the period $[t, 1]$, buys the newer version perpetual software at price p_n at time 1, and uses this software for the remaining period $[1, 1+t]$. The utility is $\int_t^1(\theta q + \alpha t - p_s)dt + (2\rho - 1)qt - p_n$.

Following a similar notion as in the baseline model, we solve this continuous user arrival model for equilibrium outcomes. The complete result derivation and proof is attached at the end of this appendix. We summarize our findings as follows.

Proposition 10 *(Equilibria with NG User Continuous Arrival) If NG users continuously arrive in the market, the SaaS quality improvement rate α affects the equilibrium outcome as follows.*

(a) (Entry Deterrence Equilibrium) If $\alpha \leq (\rho - 2\theta + 1)q$, the perpetual software vendor deters the SaaS vendor's entry into the market: The equilibrium user strategy is SP1 (Upgrade, New), where the OG users upgrade and all NG users adopt the new perpetual software. The perpetual software vendor's equilibrium prices are $p_u^* = p_n^* = (\rho - \theta)q - \frac{\alpha}{2}$.

(b) (Market Segmentation Equilibrium) If $(\rho - 2\theta + 1)q < \alpha \leq max[(2 + \sqrt{2})(\rho - \theta)q, \hat{\alpha}]$, the perpetual software vendor and the SaaS vendor segment the market: The equilibrium user strategy is SP2 (Upgrade, SaaS), where the OG users upgrade to the new perpetual software and all NG users adopt SaaS. The equilibrium prices are:

If $(\rho - 2\theta + 1)q < \alpha \leq 2(\rho - 1)q$, *then* $p_u^* = (\rho - 1)q$, $p_n^* = (\rho - 1)q$ *and* $p_s^* = (\theta - 1)q + \frac{\alpha}{2}$;

If $2(\rho - 1)q < \alpha \leq max[(2 + \sqrt{2})(\rho - \theta)q, \hat{\alpha}]$, *then* $p_u^* = (\rho - 1)q$, $p_n^* = \frac{\alpha}{2}$ *and* $p_s^* = \alpha - (\rho - \theta)q$.

(c) (Sequential Dominance Equilibrium) If $\alpha > max[(2 + \sqrt{2})(\rho - \theta)q, \hat{\alpha}]$, *the two vendors serve the market sequentially as follows: During* $[0, t_{SD}^*]$, *the perpetual software vendor serves all OG users and NG users who arrive during this interval. At* t_{SD}^*, *these users switch to the SaaS, and in addition, NG users who enter the market in the interval* $[t_{SD}^*, 1]$ *all choose SaaS during this period. The equilibrium prices are:*

$$p_u^* = \frac{(\rho-1)q[-4\alpha+(7\rho-10\theta+3)q+2\sqrt{[\alpha+(\rho-\theta)q]^2+12\alpha^2}]}{6\alpha},\ p_n^* = \frac{[-2\alpha+5(\rho-\theta)q+\sqrt{[\alpha+(\rho-\theta)q]^2+12\alpha^2}]^2}{18\alpha},\ and\ p_s^* = \frac{-2[\alpha-(\rho-\theta)q]+\sqrt{[\alpha+(\rho-\theta)q]^2+12\alpha^2}}{3}.$$

Overall, we find that all major insights under the discrete model still hold. When SaaS quality improvement is relatively small, the entry deterrence equilibrium emerges; when the SaaS quality improvement is high enough, the sequential dominance equilibrium emerges; and when the SaaS quality improvement is in the intermediate range, the market segmentation equilibrium emerges.

Moreover, we see that both vendors' optimal prices are the same as in the baseline model under the entry deterrence and market segmentation equilibria. The user groups they serve are also the same. However, the sequential dominance equilibrium is different. In the baseline model, the perpetual software vendor might charge an upgrade price that is the same as the new price, while in the continuous arrival setting, it always gives a price discount to OG users to induce them to upgrade. In addition, we also find that the perpetual vendor's new price is higher, the SaaS vendor's price is lower, and the switching time is later than the prices and switching time in the baseline model. As a result, the SaaS vendor earns a lower profit.

In summary, when the SaaS quality improvement rate is relatively high, so that sequential dominance equilibrium emerges, the perpetual software vendor is better off under the continuous arrival model. This outcome occurs mainly because NG users arrive to the market sequentially. The late arrivals are aware of the perpetual software vendor's ability to release a newer version software in the future, so they tend to choose the perpetual software upon arrival to enjoy the lower upgrade price for the future newer version.

Proofs for the Continuous User Arrival Model

Case (1) Entry Deterrence Strategy

Consider the strategy that the perpetual software vendor offers a low enough price to attract all OG users to upgrade to the new software, that NG users who arrive in the market early prefer New, and that NG users who arrive in the market late also prefer New and then upgrade to Newer at $t = 1$. Under this strategy, the SaaS vendor is out of the market, even if it offers $p_s = 0$.

First, to ensure that the OG users prefer Upgrade rather than Old, we need $\rho q - p_u \geq q$; that is, $p_u \leq (\rho - 1)q$ (K1). To ensure that the OG users prefer Upgrade rather than SaaS even if the SaaS price is 0, we need $\rho q - p_u \geq \int_0^1 (\theta q + \alpha t)dt$; that is, $p_u \leq (\rho - \theta)q - \frac{\alpha}{2}$ (K2). To ensure that NG users who arrive at $t = 0$ prefer New rather than SaaS, we need $\rho q - p_n \geq \int_0^1 (\theta q + \alpha t)dt$; that is, $p_n \leq (\rho - \theta)q - \frac{\alpha}{2}$ (K3). In addition, we also need NG users who arrive at $t = 1$ to prefer Newer rather than SaaS, so $(2\rho - 1)q - p_n \geq \theta q + \frac{3}{2}\alpha$; that is, $p_n \leq (2\rho - \theta - 1)q - \frac{3\alpha}{2}$ (K4). We can verify that both (K2) and (K3) are binding.

For NG users who arrive at $t > 0$, they might prefer New+Newer rather than New. The indifference user's entry time is determined by $\rho q - p_n = \rho q(1 - t_c) - p_n + (2\rho - 1)qt_c - p_u$; that is, $t_c = \frac{p_u}{(\rho-1)q}$. The perpetual software vendor's profit over [0,1] is $p_u + \mathrm{p}_n$. The first term is the profit from OG users, and the second term is the profit from NG users. Note that the perpetual software vendor generates the Upgrade profit from New+Newer users at $t = 1$. This profit is not counted toward the profit calculation in this software life cycle. Because the profit function increases in p_u, and note that $(\rho - 1)q > (\rho - \theta)q - \frac{\alpha}{2}$, we have $p_u^* = p_n^* = (\rho - \theta)q - \frac{\alpha}{2}$, $t_c^* = \frac{(\rho-\theta)q-\frac{\alpha}{2}}{(\rho-1)q}$, and $\pi_{perp}^{ED} = 2(\rho - \theta)q - \alpha$. Note that the condition for entry deterrence equilibrium is $\alpha \leq 2(\rho - \theta)q$.

Case (2) Market Segmentation Strategy

Consider the strategy in which the perpetual software vendor allows an SaaS vendor to enter into the market. Because OG users are more sticky than NG users, the perpetual software vendor, in giving up the NG users, charges $p_u^* = (\rho - 1)q$ to fully extract the surplus from OG users. So the perpetual software vendor serves the OG users on the interval [0,1], and the SaaS vendor serves all NG users on the interval [0,1]. Comparing this strategy with the entry deterrence strategy, the SaaS vendor charges a positive p_s.

To ensure that the OG users choose Upgrade rather than SaaS, we need $\rho q - p_u \geq \int_0^1 (\theta q + \alpha t - p_s)dt$. So $p_s \geq p_u - (\rho - \theta)q + \frac{\alpha}{2}$ (K5). Substituting p_u^{MS} into (K5), we have $p_s^* \geq (\theta - 1)q + \frac{\alpha}{2}$. To prevent the OG users from switching to SaaS during their lifetime use, we need $\theta q + \alpha t - p_s \leq \rho q$; that is, $p_s \geq \alpha - (\rho - \theta)q$ (K6). To ensure that the NG users who arrive at $t = 0$ prefer SaaS rather than New, we need $\int_0^1 (\theta q + \alpha t - p_s)dt \geq \rho q - p_n$; that is, $p_n \geq (\rho - \theta)q + p_s - \frac{\alpha}{2}$ (K7). The perpetual software vendor can price the new software at a relatively high price, such that the SaaS vendor attracts the NG users starting from time 0. Because the SaaS vendor's profit is $p_s \int_0^1 t dt$, which linearly increases in p_s, we know that (K7) is binding.

To determine p_n, we need SaaS+Newer to be preferred to SaaS; that is, $\int_0^1 (\theta q + \alpha t - p_s)dt + (2\rho - 1)qt - p_n \int_t^{1+t} (\theta q + \alpha t - p_s)dt$. So $p_n \leq [(2\rho - 1 - \theta)q - \alpha + p_s]t - \frac{\alpha}{2}t^2$. Since (K7) is binding, substituting into p_s and solving for t we have $t_c = \frac{[(\rho-1)q - \frac{\alpha}{2} + p_n] - \sqrt{[(\rho-1)q - \frac{\alpha}{2} + p_n]^2 - 2\alpha p_n}}{\alpha}$. The perpetual software vendor earns profit on the interval $[t_c, 1]$. It charges p_n as low as possible. So we have two cases: If $\alpha > 2(\rho - 1)q$, then $p_s^* = \alpha - (\rho - \theta)q$ and $p_n^* = \frac{\alpha}{2}$. The SaaS vendor's profit is $\pi_{SaaS}^{MS} = p_s^* \int_0^1 t dt = \frac{(\theta-1)q}{2} + \frac{\alpha}{4}$. If $\alpha \leq 2(\rho - 1)q$, then $p_s^* = (\theta - 1)q + \frac{\alpha}{2}$ and $p_n^* = (\rho - 1)q$. The SaaS vendor's profit is $\pi_{SaaS}^{MS} = p_s^* \int_0^1 t dt = \frac{\alpha - (\rho-\theta)q}{2}$. Under both cases, $p_u^* = (\rho - 1)q$ and $\pi_{perp}^{MS} = (\rho - 1)q$.

Case (3) Sequential Dominance Strategy

We focus on the two firms' competitive equilibrium. Assume that OG users choose Upgrade+SaaS and NG users choose New+SaaS. Again, the switching time is determined by $\theta q + \alpha t_{s3} - p_s = \rho q$; that is, $t_{s3} = \frac{p_s + (\rho-\theta)q}{\alpha}$. At $t = 0$, the OG users prefer Upgrade+SaaS rather than Upgrade if $\rho q t_{s3} - p_u + \int_{t_{s3}}^1 (\theta q + \alpha t - p_s)dt \geq \rho q - p_u$, which holds when $p_s \leq \alpha - (\rho - \theta)q$. Similarly, any NG user who arrives at time $t < t_{s3}$ prefers New+SaaS rather than New if $\rho q(t_{s3} - t) - p_n + \int_{t_{s3}}^{1+t} (\theta q + \alpha t - p_s)dt \geq \rho q - p_n$; at $t = 0$, this condition gives $p_s \leq \alpha - (\rho - \theta)q$.

The SaaS vendor's profit is expressed as $p_s(1 - t_{s3}) + p_s t_{s3}(1 - t_{s3}) + p_s \int_{t_{s3}}^1 (1 - t)dt = \frac{p_s(1-t_{s3})(3+t_{s3})}{2}$. Note that the computation of profit is different for the two groups of users. The first term is the profit from OG users who switch to SaaS at t_{s3}; the second term is the profit from the early arrival NG users (i.e., arrivals before t_{s3}) who switch to SaaS at t_{s3}; the third term is the integral of all NG users who arrive after t_{s3} so they choose SaaS directly. Solving this optimization problem we have $p_s^* = \frac{-2[\alpha-(\rho-\theta)q] + \sqrt{[\alpha+(\rho-\theta)q]^2 + 12\alpha^2}}{3}$. We can verify that p_s^* is an interior solution if $\alpha > (2 + \sqrt{2})(\rho - \theta)q$. Substituting p_s^* into the expression of t_{s3}, we get the switching time in the sequential dominance equilibrium $t_{SD}^* = \frac{-2\alpha + 5(\rho-\theta)q + \sqrt{[\alpha+(\rho-\theta)q]^2 + 12\alpha^2}}{3\alpha} < 1$. We can verify that $t_{SD}^* > 0$ under the condition $\alpha \geq (\rho - \theta)q$. At the boundary solution $p_s = \alpha - (\rho - \theta)q$, $t_{SD}^* = 1$, so (Upgrade+SaaS, New+SaaS) does not sustain as an equilibrium SP.

To ensure that OG users prefer Upgrade+SaaS rather than Old+SaaS, we need $\rho q t_{s2} - p_u + \int_{t_{s2}}^1 (\theta q + \alpha t - p_s)dt \geq q t_{s1} + \int_{t_{s1}}^1 (\theta q + \alpha t - p_s)dt$, where $t_{s2} = t_{s3}$ and $t_{s1} = \frac{p_s - (\theta-1)q}{\alpha}$ is the switching time for OG users when they choose Old+SaaS; that is, $p_u \leq \frac{(\rho-1)q[2p_s + (\rho - 2\theta + 1)q]}{2\alpha}$ (K8). Because the OG users are more sticky than the NG users, if the OG users prefer Upgrade+SaaS, then the NG users who arrive at $t = 0$ also prefer New+SaaS. Any NG user arriving before t_{s3} prefers New+SaaS rather than SaaS if $\rho q(t_{s3} - t) - p_n + \int_{t_{s3}}^{1+t} (\theta q + \alpha t - p_s)dt \geq \int_t^{1+t} (\theta q + \alpha t - p_s)dt$. Simplifying the conditions, we have $p_n \leq \frac{[(\rho-\theta)q + p_s]^2}{2\alpha}$ (K9). When $t > t_{s3}$, NG users' two strategies, SaaS+Newer and SaaS, are equivalent in the analysis because in the current planning period [0,1], the perpetual software vendor's profit for the newer version is not counted and the SaaS vendor's profit is the same.

Substituting p_s^* into (K9) we have $p_n^* = \frac{[-2\alpha + 5(\rho-\theta)q + \sqrt{[\alpha+(\rho-\theta)q]^2 + 12\alpha^2}]^2}{18\alpha}$. By (K8), $p_u^* = \frac{(\rho-1)q[-4\alpha + (7\rho - 10\theta + 3)q + 2\sqrt{[\alpha+(\rho-\theta)q]^2 + 12\alpha^2}]}{6\alpha}$. Note that the perpetual software vendor prices satisfy $p_u^* < p_n^*$. The perpetual software vendor's profit is $\pi_{perp}^{SD} = p_u^* + p_n^* t_{SD}^*$, and the SaaS vendor's profit is $\pi_{SaaS}^{SD} = \frac{p_s^*(1-t_{SD}^*)(3+t_{SD}^*)}{2}$.

In summary, the three equilibria occur in different ranges defined by α. Comparing the vendors' equilibrium profits under different α regions, we can derive the final equilibrium outcome presented in Table K1. For example, in the most complicated case, when $\alpha > (2 + \sqrt{2})(\rho -$

$\theta)q$, both market segmentation and sequential dominance are possible equilibria. Note that π_{perp}^{SD} increases in α, but π_{perp}^{MS} is independent of α. A threshold $\hat{\alpha}$ must exist such that $\pi_{perp}^{SD} > \pi_{perp}^{MS}$. Therefore, if $\alpha \geq \max[(2+\sqrt{2})(\rho-\theta)q, \hat{\alpha}]$, sequential dominance emerges as the final market equilibrium outcome.

Table K1. Equilibrium Prices and Profits Under User Continuous Arrival Model

(a) Equilibrium Prices with User Continuous Arrival

Region	p_u^*	p_n^*	p_s^*
i	$(\rho-\theta)q-\frac{\alpha}{2}$	$(\rho-\theta)q-\frac{\alpha}{2}$	0
ii	$(\rho-1)q$	$(\rho-1)q$	$(\theta-1)q+\frac{\alpha}{2}$
iii	$(\rho-1)q$	$\frac{\alpha}{2}$	$\alpha-(\rho-\theta)q$
iv	$\frac{(\rho-1)q[-4\alpha+(7\rho-10\theta+3)q+2\sqrt{[\alpha+(\rho-\theta)q]^2+12\alpha^2}]}{6\alpha}$	$\frac{[-2\alpha+5(\rho-\theta)q+\sqrt{[\alpha+(\rho-\theta)q]^2+12\alpha^2}]^2}{18\alpha}$	$\frac{-2[\alpha-(\rho-\theta)q]+\sqrt{[\alpha+(\rho-\theta)q]^2+12\alpha^2}}{3}$

(b) Equilibrium Prices with User Continuous Arrival Model

Region	Condition	Equilibrium	π_{perp}^*	π_{SaaS}^*
i	$\alpha \leq (\rho-2\theta+1)q$	Entry Deterrence	$2(\rho-\theta)q-\alpha$	0
ii	$(\rho-2\theta+1)q < \alpha \leq 2(\rho-1)q$	Market Segmentation	$(\rho-1)q$	$\frac{(\theta-1)q}{2}+\frac{\alpha}{4}$
iii	$2(\rho-1)q < \alpha \leq \max[(2+\sqrt{2})(\rho-\theta)q, \hat{\alpha}]$	Market Segmentation	$\frac{[\alpha+2(\rho-\theta)q]^2[5\alpha+4(\rho-\theta)q]}{54\alpha^2}$	$\frac{2[\alpha-(\rho-\theta)q]^2[4\alpha-(\rho-\theta)q]}{27\alpha^2}$
iv	$\alpha > \max[(2+\sqrt{2})(\rho-\theta)q, \hat{\alpha}]$	Sequential Dominance	$p_u^* + p_n^* t_{SD}^*$	$\frac{p_s^*(1-t_{SD}^*)(3-t_{SD}^*)}{2}$