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IDENTITY MANAGEMENT AND TRADABLE REPUTATION

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Appendix A

A.1 Proof of Propositions 1 and 2

We first analyze the incentive compatibility condition for low types. Notice that the objective for a low-type agent deviating to purchasing a good reputation is formulated in Equation (9). The first-order condition of Equation (9) is $\alpha w_h - 2k_l w_l' + \beta p_{gb} v_d = 0$, which leads to the deviating effort as $w_l' = w_h k_h / k_l$. The deviating profit and equilibrium profit for low types become

$$\pi_l' = k_l \frac{k_h^2}{k_l^2}\left(\beta f_h v_d\right)^2 + (1-\alpha)\beta f_h v_d + \beta\left[(1-p_{gb})v_g + p_{gb}v_b\right] - v_g$$

$$\pi_l = k_l\left(\beta f_l v_d\right)^2 + (1-\alpha)\beta f_l v_d + \beta v_b - v_b$$

Incentive compatibility for low types requires $\pi_l' \leq \pi_l$, which is equivalent to

$$k_l\left(\frac{k_h^2}{k_l^2} f_h^2 - f_l^2\right)\beta^2 v_d^2 + (1-\alpha)(f_h - f_l)\beta v_d + \left(\beta(1-p_{gb}) - 1\right)v_d \leq 0$$

Dividing both sides by v_d, we can reorganize the condition as $M_2 \beta v_d \leq k_l M_1$, where

$$M_1 = 1 - \beta\left(1 - p_{gb}\right) - (1-\alpha)\beta\left(f_h - f_l\right)$$

$$M_2 = \beta\left[\left(k_h f_h\right)^2 - \left(k_l f_l\right)^2\right]$$

Similarly, we can establish that $w_h' = w_l k_l / k_h$. The corresponding deviating profit and equilibrium profit for high types are

$$\pi_h' = k_h \frac{k_l^2}{k_h^2}\left(\beta f_l v_d\right)^2 + (1-\alpha)\beta f_l v_d + \beta v_b - v_b$$

$$\pi_h = k_h\left(\beta f_h v_d\right)^2 + (1-\alpha)\beta f_h v_d + \beta\left[(1-p_{gb})v_g + p_{gb}v_b\right] - v_g$$

Incentive compatibility condition for high types requires $\pi_h' \leq \pi_h$, which is equivalent to

$$k_h\left(f_h^2 - f_l^2 k_l^2 / k_h^2\right)\beta^2 v_d^2 + (1-\alpha)\left(f_h - f_l\right)\beta v_d + \left(\beta(1-p_{gb})-1\right)v_d \geq 0$$

Further simplify the above by dividing both sides with v_d, and we have the condition as $M_2\beta v_d \geq k_h M_1$.

Combining the two IC conditions leads to $k_h M_1 \leq M_2 \beta v_d \leq k_l M_1$. We first show that $M_1 > 0$. To see this, notice the equivalent condition is $\beta < 1/[1 - p_{gb} + (1-\alpha)(f_h - f_l)]$, where the right-hand side is greater than 1. Furthermore, the sign of M_2 is consistent with $k_h f_h - k_l f_l$. Therefore, if $k_h f_h - k_l f_l$, or, equivalently, if $p_{gb}/p_{bg} < k_l(2k_h - \alpha)/[k_h(2k_l - \alpha)]$, then $M_2 < 0$ and thus no separating equilibrium can be sustained, which proves the result in Proposition 1(b). Part (a) is a special case of Part (b) with $\alpha = 0$.

We next focus on a strict reputation system with $k_h f_h > k_l f_l$ such that both M_1 and M_2 are positive. We substitute in $v_d = \lambda_h p_{gb} / (\beta \lambda_l p_{bg} f_l + \beta \lambda_h p_{gb} f_h)$ to reorganize the IC conditions as

$$\frac{M_2}{k_l M_1} \leq \frac{\lambda_l}{\lambda_h}\frac{p_{bg}}{p_{gb}} f_l + f_h \leq \frac{M_2}{k_h M_1}$$

Define $\underline{\lambda}$ and $\overline{\lambda}$ as the values of λ_l / λ_h, that make the two IC conditions binding; that is, $\pi_l' = \pi_l$ and $\pi_h' = \pi_h$. We can derive $\underline{\lambda}$ as in Equation (12) and $\overline{\lambda}$ as in the proposition. For any separating equilibrium that satisfies the above condition, the corresponding proportion ratio λ_l / λ_h must be between $\underline{\lambda}$ and $\overline{\lambda}$; that is, $\underline{\lambda} \leq \lambda_l / \lambda_h \leq \overline{\lambda}$. Furthermore, in order to find a λ that satisfies the above conditions, it suffices to show that $(p_{gb}M_2)/(k_h M_1) \geq (p_{gb}M_1)/(k_h M_1)$ and $M_1/(k_h M_1) > f_h$. The first condition is apparently true, and the second condition can be simplified as

$$\beta > \underline{\beta} = \frac{f_h k_h}{\left[\left(k_h f_h\right)^2 - \left(k_l f_l\right)^2\right] + (1 + p_{gb}) f_h k_h + (1-\alpha)\left(f_h - f_l\right) f_h k_h}$$

In addition, because $\beta \in [0, 1]$, a separating equilibrium requires $\left[\left(k_h f_h\right)^2 - \left(k_l f_l\right)^2\right] - p_{gb} f_h k_h + (1-\alpha)\left(f_h - f_l\right) f_h k_h \geq 0$, which can be rewritten as $p_{gb} / p_{bg} \geq \underline{p}$ with $\underline{p}$ as in Equation (10).

Substituting f_θ in Equation (7) and v_d in Equation (8) into Equations (5) and (6), we can derive w_θ as in Equations (13) and (14).

A.2 Proof of Proposition 3

Notice that $\frac{2k_h - \alpha}{2k_l - \alpha} = 1 - \frac{2k_l - 2k_h}{2k_l - \alpha}$ is decreasing in α. We can verify that the second fraction in $\underline{p}$ (i.e., $\frac{2k_l^2}{\sqrt{(1-\alpha)^2 k_h^2 + 4k_l^2 k_h (1-k_h)} - (1-\alpha)k_h}$) is also decreasing in α. Therefore, $\underline{p}$ is decreasing in α.

A.3 Proof of Corollaries 1, 2, and 3

Notice that Equations (13) and (14) can be reorganized as

$$w_h = \frac{\lambda_h p_{gb}^2 (2k_l - \alpha)}{\lambda_l p_{bg}^2 (2k_h - \alpha) + \lambda_h p_{gb}^2 (2k_l - \alpha)} \text{ and } w_l = \frac{\lambda_h p_{bg} p_{gb} (2k_h - \alpha)}{\lambda_l p_{bg}^2 (2k_h - \alpha) + \lambda_h p_{gb}^2 (2k_l - \alpha)}$$

Proof of Corollary 1: Because $(2k_h - \alpha)/(2k_l - \alpha)$ is decreasing in a, it is easy to see that v_d increases in α, w_h increases in α, and w_l decreases in α.

Proof of Corollary 2: It is straightforward to verify that w_h increases in p_{gb} / p_{bg}. For w_l, we take derivative of the denominator with respect to p_{gb}/p_{bg} and obtains $(2k_l - \alpha) / (2k_h - \alpha) - \lambda_l p_{bg}^2 / (\lambda_h p_{gb}^2)$. When $p_{gb} / p_{bg} > \sqrt{(2k_h - \alpha)\lambda_l} / \sqrt{(2k_l - \alpha)\lambda_h}$, the denominator decreases and w_l increases in p_{gb} / p_{bg}; otherwise, the denominator increases and w_l decreases in p_{gb} / p_{bg}.

Proof of Corollary 3: Note that $\lambda_l / \lambda_h = \lambda_l/(1 - \lambda_l)$ is increasing in λ_l. It follows directly from the expressions of v_d, w_l, and w_h that they are all decreasing in λ_l / λ_h and λ_l.

A.4 Proof of Proposition 4

The equilibrium satisfies the following conditions: $w_b = \frac{(1-m)\lambda_h w_{hb} + (1-\lambda_h) w_l}{1-m\lambda_h}$, $w_{hb} = \frac{\alpha w_b + \beta p_{bg} v_d}{2k_h}$, $w_l = \frac{\alpha w_b + \beta p_{bg} v_d}{2k_l}$, $w_l = \frac{\beta p_{gb} v_d}{2k_h - \alpha}$, and

$$\lambda_h m = \lambda_h (1-m) w_{hb} p_{bg} + (1-\lambda_h) w_l p_{bg} + \lambda_h m \left(1 - (1 - w_h) p_{gb}\right) \tag{28}$$

The first three equations lead to

$$w_b = \frac{\beta M p_{bg} v_d}{1 - \alpha M} \tag{29}$$

$$w_{hb} = \frac{\beta p_{bg} v_d}{2k_h (1 - \alpha M)} \tag{30}$$

$$w_l = \frac{\beta p_{bg} v_d}{2k_l (1 - \alpha M)} \tag{31}$$

where $M = [(1-m)\lambda_h k_l + (1-\lambda_h) k_h] / [2(1 - m\lambda_h) k_l k_h]$.

High types at good and bad reputations must be indifferent; that is, $\pi_h = \pi_{hb}$, which leads to

$$k_h \left(w_h^2 - w_{hb}^2\right) + (1-\alpha)(w_h - w_b) + \left(\beta(1 - p_{gb}) - 1\right) v_d = 0$$

Substituting in Equations (29) to (31), v_d can be represented as

$$\beta v_d = \frac{\frac{1}{\beta} - (1-p_{gb}) - (1-\alpha)\left(\frac{p_{gb}}{2k_h-\alpha} - \frac{p_{bg}}{\frac{1}{M}-\alpha}\right)}{k_h\left[\left(\frac{p_{gb}}{2k_h-\alpha}\right)^2 - \left(\frac{p_{bg}}{2k_h(1-\alpha M)}\right)^2\right]} \tag{32}$$

The equilibrium levels of v_d and m can be solved through (28) and (32).

For low types, denote their deviating effort and profit as w_l' and π_l'. If they deviate to a good reputation, their incentive can be described through the first-order condition as $\alpha w_h - 2k_l w_l' + \beta p_{gb} v_d = 0$, which leads to $w_l' = w_h k_h / k_l$. For low types to prefer bad reputations to good reputations, we need $\pi_l' < \pi_l$, or equivalently

$$k_l(w_l'^2 - w_l^2) + (1-\alpha)(w_h - w_b) + \left(\beta(1-p_{gb}) - 1\right)v_d < 0$$

Note that $w_l' = w_h k_h / k_l$ and $w_l = w_{hb} k_h / k_l$, and $k_h / k_l < 1$. The above IC condition holds as long as $w_l' > w_l$, which can be simplified to the condition on p_{gb} / p_{bg} in Proposition 4.

We also need to ensure that $m \in [0. 1]$ in equilibrium. First, notice that both M and βv_d are decreasing in m. We then rearrange Equation (28) as

$$\lambda_h m(1-w_h)p_{gb} = \lambda_h(1-m)w_{hb}p_{bg} + (1-\lambda_h)w_l p_{bg}$$

Note that the left-hand side (LHS) of this equation is increasing in m while the right-hand side (RHS) of the equation is decreasing in m. In order for m to be between 0 and 1, it suffices to show that (1) when $m = 0$, LHS < RHS in Equation (28); and (2) when $m = 1$, LHS > RHS in Equation (28). When $m = 0$, we have $M = \lambda_h / (2k_h) + (1-\lambda_h) / (2k_l)$, LHS = 0, and RHS = $p_{bg}^2 \beta v_d (\lambda_h k_l + (1-\lambda_h)k_h) / [2k_l k_h (1-\alpha M)]$. The condition that RHS > LHS is equivalent to $\beta v_d > 0$, or $\beta < \beta_2$, where β_2 is defined as in Equation (18). Similarly, when $m = 1$, we have $M = 1/2k_l$, LHS = $\lambda_h p_{gb}\left(1 - \beta v_d p_{gb} / (2k_h - \alpha)\right)$, and RHS = $(1-\lambda_h)\beta v_d p_{bg}^2 / (2k_l - \alpha)$. The condition LHS > RHS is equivalent to $\beta v_d < \lambda_h p_{gb}(2k_h - \alpha)(2k_l - \alpha) / [\lambda_h p_{gb}^2 (2k_l - \alpha) + (1-\lambda_h) p_{bg}^2 (2k_h - \alpha)]$, or $\beta > \beta_1$, where β_1 is defined as in Equation (17). It is easy to show that $\beta_1 < \beta_2$ because

$$\frac{\lambda_h k_h\left[\left(\frac{p_{gb}}{2k_h-\alpha}\right)^2 - \left(\frac{p_{bg}}{2k_h - \frac{k_h}{k_l}\alpha}\right)^2\right]}{\frac{\lambda_h}{2k_h-\alpha} p_{gb}^2 + \frac{1-\lambda_h}{2k_l-\alpha} p_{bg}^2} > 0$$

In addition, we also need to ensure $\beta_1 < 1$, which leads to condition $\lambda_l / \lambda_h < \lambda_1$ where λ_1 is defined as in Equation (19).

A.5 Proof to Proposition 5

The equilibrium satisfies the following conditions: $w_g = \frac{\lambda_h w_h + (1-\lambda_h) n w_{lg}}{\lambda_h + (1-\lambda_h)n}$, $w_h = \frac{\alpha w_g + \beta p_{gb} v_d}{2k_h}$, $w_{lg} = \frac{\alpha w_g + \beta p_{gb} v_d}{2k_l}$, $w_l = \frac{\beta p_{bg} v_d}{2k_l - \alpha}$, and

$$(1-\lambda_h)(1-n)w_l p_{bg} = \lambda_h(1-w_h)p_{gb} + (1-\lambda_h)n(1-w_{lg})p_{gb} \tag{33}$$

The first three equations lead to

$$w_g = \frac{\beta p_{gb} v_d N}{1-\alpha N} \tag{34}$$

$$w_h = \frac{\beta p_{gb} v_d}{2k_h(1-\alpha N)} \tag{35}$$

$$w_{lg} = \frac{\beta p_{gb} v_d}{2k_l(1-\alpha N)} \tag{36}$$

where $N = [\lambda_h k_l + (1-\lambda_h) n k_h] / [2k_l k_h (\lambda_h + (1-\lambda_h) n)]$.

Low types at good and bad reputations must be indifferent; that is, $\pi_l = \pi_{lg}$, which leads to

$$k_l(w_{lg}^2 - w_l^2) + (1-\alpha)(w_g - w_l) + (\beta(1-p_{gb}) - 1)v_d = 0$$

We can solve for v_d by substituting in Equations (34) to (36) as

$$\beta v_d = \frac{-\frac{1}{\beta} + (1-p_{gb}) + (1-\alpha)\left(\frac{p_{gb}}{\frac{1}{N}-\alpha} - \frac{p_{bg}}{2k_l-\alpha}\right)}{k_l\left[\left(\frac{p_{bg}}{2k_l-\alpha}\right)^2 - \left(\frac{p_{gb}}{2k_l(1-\alpha N)}\right)^2\right]} \tag{37}$$

The equilibrium level of v_d and n are determined by (33) and (37).

For high types, denote their deviating effort and profit as w_h' and π_h'. If they deviate to a bad reputation, their incentive can be described through the first-order condition as $\alpha w_l - 2k_h w_h' + \beta p_{bg} v_d = 0$, which leads to $w_h'^2 = k_l w_l / k_h$. For high types to prefer good reputations to bad reputations, we need $\pi_h' < \pi_h$, or equivalently

$$k_h(w_h'^2 - w_h^2) + (1-\alpha)(w_l - w_g) - (\beta(1-p_{gb}) - 1)v_d < 0$$

Note that $w_h' = k_l w_l / k_h$ and $w_h = k_l w_{lg} / k_h$, and $k_l / k_h > 1$. The above IC condition holds as long as $w_h' > w_h$, which can be simplified to the condition on p_{gb}/p_{bg} in Proposition 5.

We also need to ensure that in equilibrium $n \in [0, 1]$. First, notice that both N and βv_d are decreasing in n, which can be verified with simple algebra. We then rearrange Equation (33) as

$$(1-\lambda_h)(1-n) w_l p_{bg} = \lambda_h (1-w_h) p_{gb} + (1-\lambda_h) n (1-w_{\mathrm{lg}}) p_{gb} \tag{38}$$

Notice that the left-hand side of this equation is decreasing in n while the right-hand side is increasing in n. For the equilibrium n to be between 0 and 1, it must satisfy two conditions: (1) when $n = 0$, Equation (38) becomes an inequality with the left-hand side (LHS) greater than the right-hand side (RHS); and (2) when $n = 1$, Equation (38) becomes an inequality with the left-hand side less than the right-hand side. At $n = 0$, $N = 1/(2k_h)$, LHS = $(1-\lambda_h)\beta v_d p_{bg}^2 / (2k_l - \alpha)$, and RHS = $\lambda_h p_{gb}(2k_h - \alpha - \beta p_{gb} v_d) / (2k_h - \alpha)$. The equivalent condition of LHS > RHS is

$$\beta v_d = \frac{-\frac{1}{\beta} + (1-p_{gb}) + (1-\alpha)\left(\frac{p_{gb}}{2k_h-\alpha} - \frac{p_{bg}}{2k_l-\alpha}\right)}{k_l\left[\left(\frac{p_{bg}}{2k_l-\alpha}\right)^2 - \left(\frac{p_{gb}}{2k_l-\frac{k_l}{k_h}\alpha}\right)^2\right]} > \frac{\lambda_h p_{gb}}{\frac{(1-\lambda_h)p_{bg}^2}{2k_l-\alpha} + \frac{\lambda_h p_{gb}^2}{2k_h-\alpha}}$$

Similarly, when $n = 1$, then $N = \frac{\lambda_h}{2k_h} + \frac{1-\lambda_h}{2k_l}$, $LHS = 0$, and $RHS > 0$ is equivalent to

$$\beta v_d = \frac{-\frac{1}{\beta} + \left(1 - p_{gb}\right) + \left(1-\alpha\right)\left(\frac{p_{gb}}{\frac{1}{N}-\alpha} - \frac{p_{bg}}{2k_l - \alpha}\right)}{k_l\left[\left(\frac{p_{bg}}{2k_l-\alpha}\right)^2 - \left(\frac{p_{gb}}{2k_l(1-\alpha N)}\right)^2\right]} < \frac{1}{\left(\frac{\lambda_h}{2k_h(1-\alpha N)} + \frac{1-\lambda_h}{2k_l(1-\alpha N)}\right)p_{gb}}$$

The corresponding condition in terms of β becomes $\beta_3 < \beta < \beta_4$, where β_3 and β_4 are defined as in Equations (20) and (21). We can verify that $\beta_3 < \beta_4$. In addition, we also need to ensure $\beta_3 < 1$ and $\beta_4 > 0$, for which it suffices to show that $1 / \beta_3 > 1$, or equivalently, $\lambda_l / \lambda_h > \lambda_2$ where λ_2 is defined as in Equation (22).

A.6 Proof of Proposition 6

Consider an equilibrium where (1) low types own reputation $j = 0$, and high types have all the rest; (2) the reputation value difference is such that $v_1 - v_0 = \alpha$, and $v_j - v_{j-1} = b$ for $j \geq 2$. The corresponding equilibrium efforts and profits are

$$\begin{aligned} w_0 &= \frac{a}{2k_l - \alpha} \\ \pi_0 &= k_l w_0^2 + (1-\alpha) w_0 \\ w_1 &= \frac{a+b}{2k_h - \alpha} \\ \pi_1 &= k_h w_1^2 + (1-\alpha) w_1 - a \\ w_{j\geq 2} &= \frac{2b}{2k_h - \alpha} \\ \pi_{j\geq 2} &= k_h w_j^2 + (1-\alpha) w_j - b \end{aligned} \tag{39}$$

The separating equilibrium has to satisfy the following five conditions.

Condition 1: $\pi_1 = \pi_{j \geq 2}$, which leads to

$$a + 3b = \frac{(2k_h - 1)(2k_h - \alpha)}{k_h} \tag{40}$$

Condition 2: the proportion of reputation 0 is $\lambda_0 = 1 - \lambda_h$.

Condition 3: the proportion of all the other reputations are such that $\lambda_1 = 2(2k_h - \alpha)\lambda_0 / [(2k_l - \alpha)(2k_h - \alpha) - (a+b)]$, $\lambda_2 = (a+b)\lambda_1 / (2k_h - \alpha - 2b)$, and $\lambda_{j\geq 3} = 2b\lambda_{j-1} / (2k_h - \alpha - 2b)$. The aggregate of these proportions must be λ_h; that is,

$$\frac{2k_h - \alpha}{2k_l - \alpha} \frac{2}{(2k_h - \alpha) - (a+b)} \frac{2k_h - \alpha + a - 3b}{2k_h - \alpha - 4b} = \frac{\lambda_h}{1 - \lambda_h} \tag{41}$$

Condition 4 (ICH): high types prefer $j \geq 1$ to 0; that is,

$$k_h\left(\frac{2b}{2k_h - \alpha}\right)^2 + (1-\alpha)\frac{2b}{2k_h - \alpha} - b \geq \frac{k_l}{k_h}\left[k_l\left(\frac{a}{2k_l - \alpha}\right)^2 + (1-\alpha)\frac{a}{2k_l - \alpha}\right] \tag{42}$$

This is equivalent to $\pi_h \geq \pi_l k_l / k_h$.

Condition 5 (ICL): low types prefer 0 to $j \geq 1$; that is,

$$k_l\left(\frac{a}{2k_l-\alpha}\right)^2+(1-\alpha)\frac{a}{2k_l-\alpha}\geq\frac{k_h}{k_l}\left[k_h\left(\frac{2b}{2k_h-\alpha}\right)^2+(1-\alpha)\frac{2b}{2k_h-\alpha}\right]-b \tag{43}$$

which is equivalent to $\pi_l \geq \pi_h k_h / k_l - b(k_l - k_h) / k_l$.

We prove the existence of such an equilibrium in two steps.

Step 1: We want to show that Equation (41) simply moves the equilibrium along the line described by Equation (40). In other words, as $\lambda_h / (1-\lambda_h)$ varies from 0 to ∞, the corresponding variations of a and b cover every single point in Equation (40).

First, substitute Equation (40) into Equation (41) and rewrite the latter as a function of b only:

$$\lambda=\frac{\lambda_h}{1-\lambda_h}=\frac{M-3b}{N-4b}\frac{N+M-6b}{N-M+2b}$$

where $M = (2k_h - \alpha)(2k_h - 1) / k_h$, and $N = 2k_h - \alpha$. Notice that

$$\frac{\partial\lambda}{\partial b}=\frac{60Nb^2+(36N^2-20MN+16M^2)b+(7M^2N-4MN^2-3N^3-4M^3)}{(N-4b)^2(N-M+2b)^2}$$

and $60N > 0$, $36N^2 - 20MN + 16M^2 > 0$, and $7M^2N - 4MN^2 - 3N^3 - 4M^3 < 0$. Hence, λ decreases in b when b is small, and increases in b when b is large.

We also need to check whether Equation (41) itself imposes any restrictions on the variations of b. The only condition that has to be satisfied is $\lambda \geq 0$. Note that, $2k_h - \alpha - (a + b) > 0$ is equivalent to $2b > (2k_h - \alpha)(2k_h - 1)/k_h$, which is always true because $k_h - 1 < 0$. Similarly, $2k_h - \alpha - 3b + \alpha \geq 0$ leads to $2a \geq (2k_h - \alpha)(2k_h - 1)/k_h$, which is also true for all $a \geq 0$. Finally, $2k_h - \alpha - 4b > 0$ leads to $b < (2k_h - \alpha)/4$, which is equivalent to $a \geq (2k_h - \alpha)(5k_h - 4)/(4k_h)$ because $b = (2k_h - \alpha)(2k_h - 1)/(3k_h) - a/3$. Note that, if $k_h \leq 4/5$, the inequality always holds. Otherwise, Equation (41) imposes the additional condition that $b > (2k_h - \alpha)/4$.

So, when $k_h \leq 4/5$, λ first decreases and then increases in b, as b goes up. In this case, every point on the line in (40) can be reached at a certain λ, and we only need to find one point on (40) to demonstrate the existence of separation. When $k_h > 4/5$, λ first decreases and then increases in b, and goes to infinity as b approaches $(2k_h - \alpha)/4$. In this case, Equation (41) covers only part of the line in (40); that is, $0 \leq b < (2k_h - \alpha)/4$. We need to find a point within this range that satisfies the ICs to show the existence of separation.

Step 2: Reorganizing the ICH and ICL by substituting in $a = M - 3b$, we have

$$k_h b\leq\left(\frac{2bk_h}{2k_h-\alpha}\right)^2-\left(\frac{(M-3b)k_l}{2k_l-\alpha}\right)^2+(1-\alpha)\left(\frac{2bk_h}{2k_h-\alpha}-\frac{(M-3b)k_l}{2k_l-\alpha}\right)\leq k_l b$$

We denote the middle expression as $T(b)$. Notice that

$$\frac{\partial T}{\partial b}=\left(\frac{k_h}{2k_h-\alpha}\right)^2 8b+\left(\frac{k_l}{2k_l-\alpha}\right)^2 6(M-3b)+(1-\alpha)\left(\frac{2k_h}{2k_h-\alpha}-\frac{3k_l}{2k_l-\alpha}\right)>0$$

It is easy to see $T(0) < 0$, and $T(M/3) > 0$. Hence, as long as $T(b = M/3) \geq k_h b$ when $k_h \leq 4/5$ or $T(b = (2k_h - \alpha)/4) > k_h b$ when $k_h > 4/5$, a separation must exist. When $k_h \leq 4/5$, substituting in $b = M/3$, we need $\left(\frac{2bk_h}{2k_h-\alpha}\right)^2+(1-\alpha)\frac{2bk_h}{2k_h-\alpha}-k_h b\geq 0$, which leads to $k_h \geq 3\alpha/2 - 1$.

In order for $0 \leq k_h \leq 4/5$, α has to satisfy that $2/3 \leq \alpha \leq 1$. Therefore, when $2/3 \leq \alpha \leq 1$ and $3\alpha/2 - 1 \leq k_h \leq 4/5$, there exists a separating equilibrium.