THE "INVISIBLE HAND" OF PIRACY: AN ECONOMIC ANALYSIS OF THE INFORMATION-GOODS SUPPLY CHAIN

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Appendix A

Technical Details of All the Extensions

In this appendix, we provide some of the technical details that were omitted from the main paper for readability.

Heterogeneity in Piracy Cost

Given the demand function in (3), we can solve the retailer's maximization problem $\max_p (p-w)q(p)$

$$p(w) = \begin{cases} p_0(w) = \frac{1+w}{2}, & \text{Case R0} \\ p_1(w) = \frac{1-\beta+r(1-\alpha)+w(1-\alpha\beta)}{2(1-\alpha\beta)} & \text{Case R1} \\ p_2(w) = \frac{1+w}{2}, & \text{Case R2} \\ p_3(w) = \frac{1-\beta+r(1-\alpha)+\alpha(g+hr)+w}{2}, & \text{Case R3} \\ p_4(w) = \frac{1-\beta+g+hr+w}{2}, & \text{Case R4} \end{cases}$$

For each case, we can now solve the manufacturer's profit maximization problem $\max_{w_i} w_i p_i(w_i)$ to obtain the wholesale price, which can be substituted above to get the retail price. The overall solution for each case can be written as

$$(w,p) = \begin{cases} (w_0, p_0) = \left(\frac{1}{2}, \frac{3}{4}\right), & \text{Case R0} \\ (w_1, p_1) = \left(\frac{1-\beta+r(1-\alpha)}{2(1-\alpha\beta)}, \frac{3(1-\beta+r(1-\alpha))}{4(1-\alpha\beta)}\right), & \text{Case R1} \\ (w_2, p_2) = \left(\frac{1}{2}, \frac{3}{4}\right), & \text{Case R2} \\ (w_3, p_3) = \left(\frac{1-\beta+r(1-\alpha)+\alpha(g+hr)}{2}, \frac{3(1-\beta+r(1-\alpha)+\alpha(g+hr))}{4}\right), & \text{Case R3} \\ (w_4, p_4) = \left(\frac{1-\beta+g+hr}{2}, \frac{3(1-\beta+g+hr)}{4}\right), & \text{Case R4} \end{cases}$$

We now turn our attention to the limit regions:

- **Case R1A:** In this region, the retailer is forced to set $p_{1\text{A}} = \frac{r}{\beta}$. The wholesale price in this case is $w_{1\text{A}} = \frac{r}{\beta} - \frac{(1-\beta)(\beta-r)}{\beta(1-\alpha\beta)}$, which can be found by simply equating $p_1(w)$ to $p_{1\text{A}}$ and solving for w.
- **Case R1B:** In this region, the retailer must set $p_{1\text{B}} = 1-\beta+r$. The wholesale price in this case is given by $w_{1\text{B}} = 2(1-\beta) + r - \frac{(1-\beta)(1-r\alpha)}{1-\alpha\beta}$, which is the solution of $p_1(w) = p_{1\text{B}}$.
- **Case R3A:** In this region, too, the retailer is forced to set $p_{3\text{A}} = 1-\beta+r$. The corresponding wholesale price is obtained from the solution of $p_3(w) = p_{3\text{A}}$ and is given by $w_{3\text{A}} = 1-\beta+r-\alpha(g+hr-r)$.
- **Case R3B:** The limit retail price in this case is given by $p_{3\text{B}} = \frac{g+hr}{\beta}$. The corresponding wholesale price is obtained from the solution of $p_3(w) = p_{3\text{B}}$ and is given by $w_{3\text{B}} = \frac{(2-\alpha\beta)(g+hr)}{\beta} - (1-\beta+r(1-\alpha))$. In this case, a valid retail price must satisfy $\frac{p_{3\text{B}}-r}{1-\beta} \leq 1$.
- **Cases R4A and R4B:** In these cases as well, the retailers is forced to set a limit retail price of $p_{4\text{A}} = p_{4\text{B}} = \frac{g+hr}{\beta}$. The wholesale price in R4A is obtained from the solution of $p_3(w) = p_{4\text{A}}$ and is given by $w_{4\text{A}} = \frac{(2-\alpha\beta)(g+hr)}{\beta} - (1-\beta+r(1-\alpha))$, the only difference with R3B being that, now, $\frac{p_{4\text{A}}-r}{1-\beta} > 1$. Case 4A must also satisfy $w_{4\text{A}} < p_{4\text{A}}$. When this is violated, we enter Case 4B as another limit case, where $w_{4\text{A}} = p_{4\text{A}} = \frac{g+hr}{\beta}$.

With these closed form solutions for wholesale and retail prices, it is easy to find the manufacturer's and retailer's profits as

$$(\pi_m, \pi_r) = \begin{cases} (\pi_{m0}, \pi_{r0}) = \left(\frac{1}{8}, \frac{1}{16}\right), & \text{Case R0} \\ (\pi_{m1}, \pi_{r1}) = \left(\frac{\left(1-\beta+r(1-\alpha)\right)^2}{8(1-\beta)(1-\alpha\beta)}, \frac{\left(1-\beta+r(1-\alpha)\right)^2}{16(1-\beta)(1-\alpha\beta)}\right), & \text{Case R1} \\ (\pi_{m1\text{A}}, \pi_{r1\text{A}}) = \left(\frac{(\beta-r)\left(r\left(2-\beta(1+\alpha)\right)-\beta(1-\beta)\right)}{\beta^2(1-\alpha\beta)}, \frac{(\beta-r)^2(1-\beta)}{\beta^2(1-\alpha\beta)}\right), & \text{Case R1A} \\ (\pi_{m1\text{B}}, \pi_{r1\text{B}}) = \left(\frac{\alpha(\beta-r)\left(1-\beta+r(1+\alpha)-2\alpha\beta(1-\beta+r)\right)}{1-\alpha\beta}, \frac{\alpha^2(\beta-r)^2(1-\beta)}{1-\alpha\beta}\right), & \text{Case R1B} \\ (\pi_{m2}, \pi_{r2}) = \left(\frac{\alpha}{8}, \frac{\alpha}{16}\right), & \text{Case R2} \\ (\pi_{m3}, \pi_{r3}) = \left(\frac{\left(1-\beta+r(1-\alpha)+\alpha(g+hr)\right)^2}{8(1-\beta)}, \frac{\left(1-\beta+r(1-\alpha)+\alpha(g+hr)\right)^2}{16(1-\beta)}\right), & \text{Case R3} \\ (\pi_{m3\text{A}}, \pi_{r3\text{A}}) = \left(\frac{\alpha(g+hr-r)\left(1-\beta-\alpha(g+hr)+r(1+\alpha)\right)}{1-\beta}, \frac{\alpha^2(g+hr-r)^2}{1-\beta}\right), & \text{Case R3A} \\ (\pi_{m3\text{B}}, \pi_{r3\text{B}}), & \text{Case R3B} \\ (\pi_{m4}, \pi_{r4}) = \left(\frac{\alpha(1-\beta+g+hr)^2}{8(1-\beta)}, \frac{\alpha(1-\beta+g+hr)^2}{16(1-\beta)}\right), & \text{Case R4} \\ (\pi_{m4\text{A}}, \pi_{r4\text{A}}), & \text{Case R4A} \\ (\pi_{m4\text{B}}, \pi_{r4\text{B}}) = \left(\frac{\alpha(g+hr)(\beta-g-hr)}{\beta^2}, 0\right), & \text{Case R4B} \end{cases}$$

where

$$\pi_{m3\text{B}} = \frac{(2(g+hr)-\beta(1-\beta+r(1-\alpha)+\alpha(g+hr)))(\beta\left(1-\beta+r(1-\alpha)\right)-(g+hr)(1-\alpha\beta))}{\beta^2(1-\beta)},$$

$$\pi_{r3\text{B}} = \frac{(g+hr-\beta(1-\beta+r(1-\alpha)+\alpha(g+hr)))^2}{\beta^2(1-\beta)},$$

$$\pi_{m4\text{A}} = \frac{\alpha(g+hr-\beta)(\beta\left(1-\beta+r(1-\alpha)+\alpha(g+hr)\right)-2(g+hr))}{\beta^2}, \quad \text{and}$$

$$\pi_{r4\text{A}} = \frac{\alpha(g+hr-\beta)((1-\alpha\beta)(g+hr)-\beta(1-\beta+r(1-\alpha)))}{\beta^2}.$$

The boundaries between these regions are obtained in two steps. First, we apply the validity conditions in (3) to R0, R1, R2, R3, and R4. We also apply the appropriate validity conditions to all the six limit regions. Once we have curtailed these individual regions by their validity conditions, only a few overlapping regions remain. To determine their explicit boundaries, we then compare the manufacturer's profits across those overlapping cases. Because all our price and profit expressions are in closed form, we can easily find these boundaries in closed form as well. Once we curtail the overlapping regions using these boundaries, we get a unique equilibrium solution for every point in the parameter space. We omit the cumbersome algebraic expressions in favor of plots of the manufacturer's and retailer's profits as functions of r and α; Figure A1 shows these profit plots; for these plots, $\beta = 0.75$, and the heterogeneity level is moderate ($g = 0.1$ and $h = 2$). It is comforting to see that a two-dimensional slice of these plots for very small α-values mimic our results depicted in Figure 2(a).

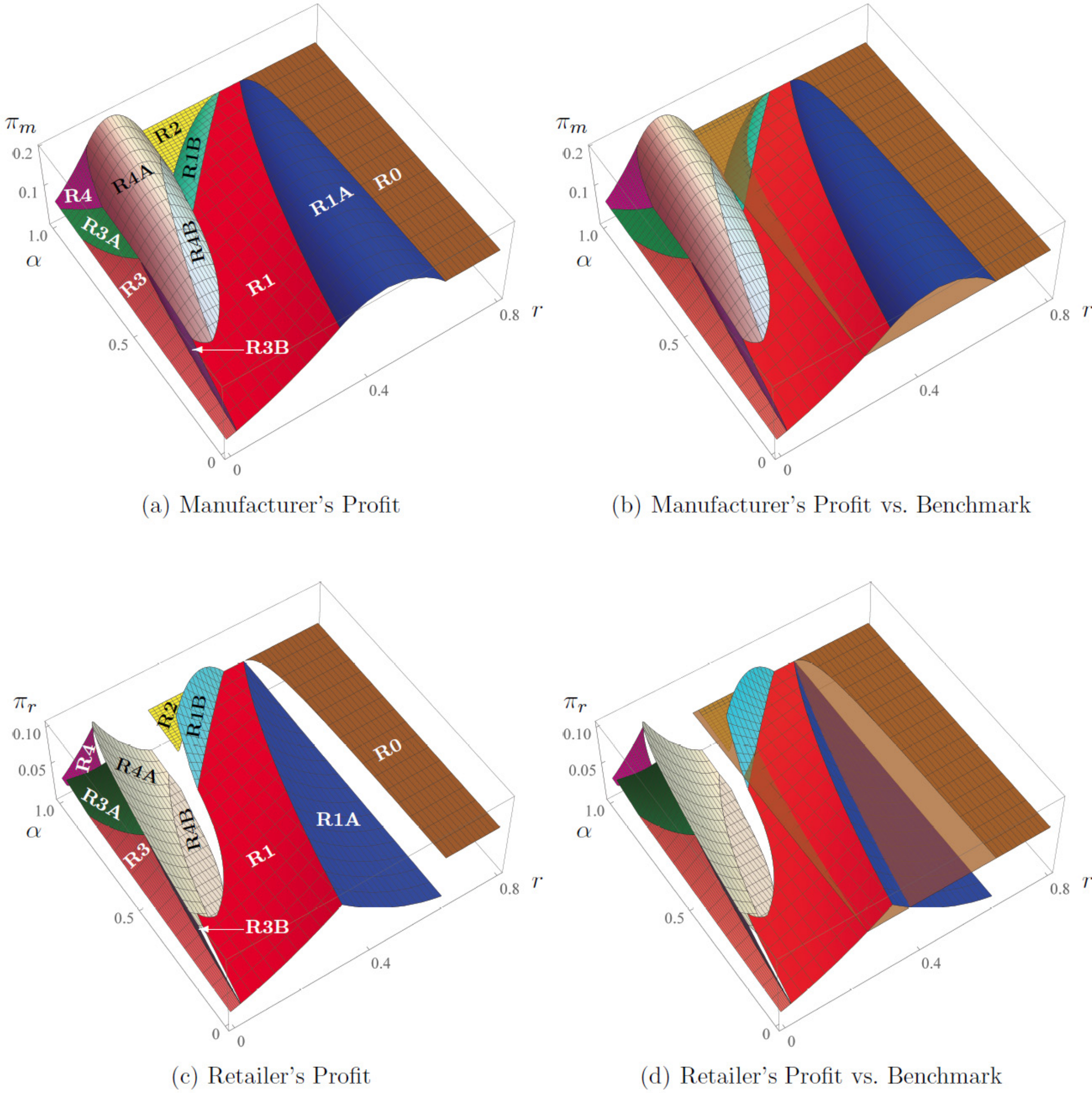


Figure A1. Profit as a Function of r and α; $\boldsymbol{\beta = 0.75, g = 0.1, h = 2}$

A careful observation of the plots in Figure A1(b) and (d) reveals that there is indeed a region spanning portions of R1 and R1A, where both the manufacturer and retailer have profits higher than their respective benchmark values in R0. In fact, the red-blue humps in both plots over the translucent R0-plane are clearly visible. This win-win region is denoted by $(\tilde{\rho}_3, \tilde{\rho}_4)$ in Figure 7; $\tilde{\rho}_3$ is obtained by comparing π_{m1} and $\pi_{m1\mathrm{B}}$ with π_{m0}, and $\tilde{\rho}_4$, by comparing $\pi_{r1\mathrm{A}}$ with π_{r0}. We find

$$\tilde{\rho}_3 = \begin{cases} \dfrac{\sqrt{(1-\beta)(1-\alpha\beta)}-(1-\beta)}{1-\alpha}, & \text{if } \alpha \le \frac{7-8\beta+\sqrt{49-48\beta}}{32\beta(1-\beta)} \\ \dfrac{\sqrt{2\alpha(1-\alpha)(1-\alpha\beta)(2\alpha\beta-1)}}{4\alpha(\alpha(2\beta-1)-1)} + \dfrac{1-2\beta-3\alpha\beta+4\alpha\beta^2}{2(\alpha(2\beta-1)-1)}, & \text{otherwise} \end{cases}$$

and

$$\tilde{\rho}_4 = \beta\left(1 - \frac{1}{4}\sqrt{\frac{1-\alpha\beta}{1-\beta}}\right)$$

A point to note here is that the above thresholds are independent of both g and h, and depend only on α, the fraction of the high type. Furthermore, in the case of no heterogeneity, that is, when $\alpha \to 0$, they reduce to the original (ρ_3, ρ_4) window

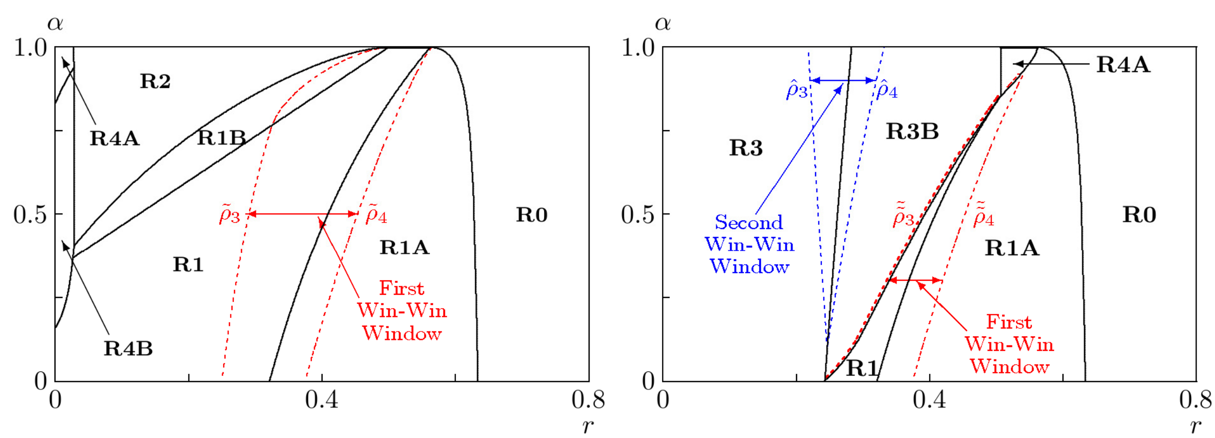
$$\lim_{\alpha\to 0} \tilde{\rho}_3 = \rho_3 \quad \text{and} \quad \lim_{\alpha\to 0} \tilde{\rho}_4 = \rho_4.$$

A structural observation is now in order. There are essentially two levers that control heterogeneity in the piracy cost. The first lever, α, which simply indicates the *extent of heterogeneity*, exhibits a behavior that is essentially the same at both the extremes. When α is small, we get back our original situation, because the fraction of the high type is negligible, making heterogeneity disappear for all practical purposes.

However, the same is also true for very high α, in which case, the fraction of the low type is negligible, and we get back our original problem with a linearly transformed piracy cost. This inherent symmetry of the setup is quite important to fully grasp this complicated analysis. Now, while α indicates the extent, the *level of heterogeneity* is determined by the second lever of the (g, h) pair—when g and h are high, either individually or together, heterogeneity is high, but, when they are both small, that is, when $g \to 0$ and $h \to 1$, heterogeneity once again disappears, and we get back to our original problem setting.

Now, even though the $(\tilde{\rho}_3, \tilde{\rho}_4)$ window is independent of g and h, we are still not assured of the existence of a win-win window. To fully understand the impact of g and h on the existence of the win-win window, we need to determine what happens when they move from their moderate values of $g = 0.1$ and $h = 2$ as reported in Figure A1. It turns out that the $\tilde{\rho}_3$-threshold, which was obtained by comparing π_{m1} and π_{m1B} with π_{m0}, may no longer provide the valid left limit of the win-win window, if boundaries of R1 and R1B encroach upon $\tilde{\rho}_3$.

When g or h increases from its moderate value, there are no problems with the win-win window represented by $(\tilde{\rho}_3, \tilde{\rho}_4)$. This is because the regions to the left of R1 and R1B actually move further to the left when either g or h increases. Therefore, there is no encroaching on $\tilde{\rho}_3$, and the win-win window derived above remains intact. This is clearly visible in Figure A2(a).



(a) High Heterogeneity: g=0.5, h=5 (b) Low Heterogeneity: g=0.01 and h=1.1

Figure A2. Partitions of the (r, α) Space for Extreme g and h; $\beta = 0.75$

However, as both g and h become small, the regions to the left of R1 and R1B start moving in towards the right, squeezing R1 and R1B in the process. At some point, when g and h are both really small, the boundary between R1 and R4B moves in sufficiently to encroach on the $\tilde{\rho}_3$-threshold; see Figure A2(b). When that happens, $(\tilde{\rho}_3, \tilde{\rho}_4)$ is no longer the valid win-win window. The correct one becomes $(\tilde{\tilde{\rho}}_3, \tilde{\tilde{\rho}}_4)$

$$\tilde{\tilde{\rho}}_3 = \max\{\tilde{\rho}_3, b_1, b_2, b_3\} \quad \text{and} \quad \tilde{\tilde{\rho}}_4 = \max\{\tilde{\rho}_4, b_3\}$$

where b_1 is the boundary between regions R1 and R4B, b_2 between R1 and R3B, and b_3 between R1A and R4A. When g and h are very small, all these boundaries, b_1, b_2, and b_3, get pushed to the right, resulting in some shrinkage of the win-win window, $(\tilde{\tilde{\rho}}_3, \tilde{\tilde{\rho}}_4)$. However, well before this win-win window can be fully usurped, a second win-win window starts appearing to its left. The emergence of this second win-win window may seem surprising at first, but can be clearly predicted from the symmetry of the problem we discussed earlier. The first win-win window, $(\tilde{\tilde{\rho}}_3, \tilde{\tilde{\rho}}_4)$, occurs because of the existence of the low type. When the *level* of heterogeneity is low, that is, both g and h are small, the high type is now very close to the low type and must, therefore, behave in a similar fashion, implying that the high type ought to get a win-win window of its own.

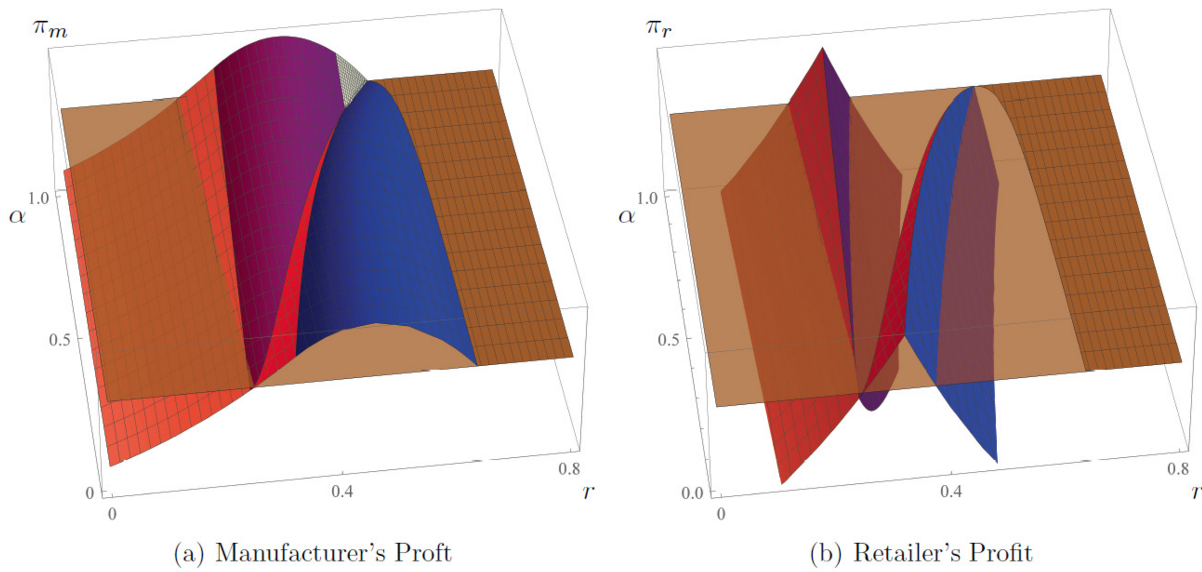


Figure A3. Profit as a Function of r and α; $\beta = 0.75, g = 0.01, h = 1.1$

To illustrate, we once again plot the manufacturer's and retailer's profits in Figure A3, this time for $g = 0.01$ and $h = 1.1$. Figure A3 clearly reveals the pink-purple humps above the benchmark levels in both the profit plots; of course, these humps are there in addition to the original red-blue ones, which have now shrunk somewhat. This second win-win window is denoted $(\hat{\rho}_3, \hat{\rho}_4)$; $\hat{\rho}_3$ and $\hat{\rho}_4$ can be easily obtained by comparing the retailer's profit in regions R3 and R3B with their benchmark value in R0. We get

$$\hat{\rho}_3 = \frac{\sqrt{1-\beta}-(1-\beta+g\alpha)}{1+\alpha(h-1)} \quad \text{and} \quad \hat{\rho}_4 = \frac{\beta\left(4(1-\beta)-\sqrt{1-\beta}\right)-4g(1-\alpha\beta)}{4(h(1-\alpha\beta)-\beta(1-\alpha))}$$

As g and h decrease even further, the first window, $(\tilde{\tilde{\rho}}_3, \tilde{\tilde{\rho}}_4)$, shrinks, but the second window, $(\hat{\rho}_3, \hat{\rho}_4)$, actually expands. It is easy to see that, when heterogeneity is absent, the second window becomes the same as the original (ρ_3, ρ_4) window, because

$$\lim_{\substack{g\to 0\\ h\to 1}} \hat{\rho}_3 = \rho_3 \quad \text{and} \quad \lim_{\substack{g\to 0\\ h\to 1}} \hat{\rho}_4 = \rho_4$$

Commercial Pirates

In this setup, a consumer can enjoy a utility of $(v-p)$ from purchasing the legal version, or $(v\beta - r - s)$ from a pirated copy. Similar to (1), the legal and illegal demands for given p and s, respectively denoted $q(p,s)$ and $\bar{q}(p,s)$, can now be rewritten as

$$q(p) = \begin{cases} 1 - \frac{p-(r+s)}{1-\beta}, & \text{if } p > \frac{r+s}{\beta} \\ 1-p, & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{q}(p) = \begin{cases} \frac{p-(r+s)}{1-\beta} - \frac{r+s}{\beta}, & \text{if } p > \frac{r+s}{\beta} \\ 0, & \text{otherwise} \end{cases} \tag{A1}$$

Given these demand functions, the commercial pirate chooses s in order to maximize its profit $\pi_s(s) = s\,\bar{q}(p,s)$. Since $\frac{\partial^2 \pi_s}{\partial s^2} = -\frac{2}{\beta(1-\beta)} < 0$, we solve the first order condition, $\frac{\partial \pi_s}{\partial s} = \frac{p\beta - r - 2s}{\beta(1-\beta)} = 0$, to obtain the optimal s for a given p

$$s^*(p) = \begin{cases} \frac{p\beta - r}{2}, & \text{if } p > \frac{r}{\beta} \\ 0, & \text{otherwise} \end{cases} \tag{A2}$$

Anticipating this response from the commercial pirate, the retailer chooses p in order to maximize its profit $\pi_r(p) = (p-w)\, q(p, s^*(p))$. Now, if $q(p,s) = 1 - \frac{p-(r+s)}{1-\beta}$, then $\pi_r(p,s) = (p-w)\left(1 - \frac{p-(r+s)}{1-\beta}\right)$. Substituting s for $s^*(p)$ in (A2) and taking the derivative with respect to p, we obtain

$$\frac{\partial \pi_r}{\partial p} = \frac{2+r+2w-2p(2-\beta)-\beta(2+w)}{2(1-\beta)} \tag{A3}$$

Since $\frac{\partial^2 \pi_r}{\partial p^2} = -1 - \frac{1}{1-\beta} < 0$, the first-order condition results in $p^*(w) = \frac{r+2(1+w)-\beta(2+w)}{2(2-\beta)}$, which, according to (A1), must be greater than $\frac{r+s^*(p^*(w))}{\beta}$, or $w > \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$, for this solution to be valid.

If, on the other hand, $q(p,s) = 1-p$, then $\pi_r(p,s) = \pi_r(p) = (p-w)(1-p)$. Therefore, we get

$$\frac{\partial \pi_r}{\partial p} = 1 - 2p + w \tag{A4}$$

Since the second-order condition is trivially satisfied, we can equate (A4) to zero to obtain $p^*(w) = \frac{1+w}{2}$, which must be smaller than $\frac{r+s^*(p^*(w))}{\beta}$, or $w < \frac{2r}{\beta} - 1$, for this solution to be valid.

Now, for moderate values of w, that is, if $\frac{2r}{\beta} - 1 \le w \le \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$, $\frac{\partial \pi_r}{\partial p}$ given by (A3) is negative, whereas that given by (A4) is positive. Naturally, the optimal p is simply $\frac{r}{\beta}$. Taken together, the optimal retail price for a given w, $p^*(w)$, is

$$p^*(w) = \begin{cases} \frac{r+2(1+w)-\beta(2+w)}{2(2-\beta)}, & \text{if } w > \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta} \\ \frac{r}{\beta}, & \text{if } \frac{2r}{\beta} - 1 \le w \le \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta} \\ \frac{1+w}{2}, & \text{otherwise} \end{cases} \tag{A5}$$

The manufacturer, the first mover in the game, anticipates the retailer's pricing decisions and chooses the optimal wholesale price w^* to maximize $\pi_m(w) = w\, q(p^*(w), s^*(p^*(w)))$. It is clear from (A5) that we have three cases to consider: (i) $w > \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$, (ii) $\frac{2r}{\beta} - 1 \le w \le \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$, and (iii) $w < \frac{2r}{\beta} - 1$.

For case (i), the manufacturer's profit is $\pi_m = \frac{w(2+r-2\beta-w(2-\beta))}{4(1-\beta)}$. Since $\frac{\partial^2 \pi_m}{\partial w^2} = -\frac{2-\beta}{2(1-\beta)} < 0$, the first order condition, $\frac{\partial \pi_m}{\partial w} = \frac{2+r-2\beta-2w(2-\beta)}{4(1-\beta)} = 0$, results in $w^* = \frac{2+r-2\beta}{2(2-\beta)}$, which, according to (A5), must be greater than $\frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$, or $r \le \frac{6\beta(1-\beta)}{8-7\beta} = \tilde{\rho}_1$, for this equilibrium to be valid.

For case (ii), $p^* = \frac{r}{\beta}$. The manufacturer, unwilling to leave money on the table, always chooses the highest value from the range $\frac{2r}{\beta} - 1 \le w \le \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$, resulting in $w^* = \frac{2r}{\beta} - 1 + \frac{\beta-r}{2-\beta}$. This equilibrium is valid across all $r \le \beta$ since $s^* = 0$ in this equilibrium. If $r > \beta$, then $p^* = \frac{r}{\beta} > 1$, and no consumer would buy the product. Therefore, $r > \beta$ cannot happen in case (ii).

Finally, in case (iii), $p^*(w) = \frac{1+w}{2}$, and the manufacturer's profit is $\pi_m = \frac{w(1-w)}{2}$, implying $w^* = \frac{1}{2}$. According to (A5), this w^* must be no more than $\frac{2r}{\beta} - 1$, implying $r \ge \frac{3\beta}{4} = \tilde{\rho}_5$. It is easy to verify that $\tilde{\rho}_1 < \tilde{\rho}_5$.

Now, case (ii) is the only valid equilibrium if $\tilde{\rho}_1 \le r \le \tilde{\rho}_5$. On the other hand, if $r < \tilde{\rho}_1$, both cases (i) and (ii) are valid. However, the optimal profit from an interior solution ought to be higher, which immediately implies that case (i) is the equilibrium outcome for $r < \tilde{\rho}_1$. Further, if $r > \tilde{\rho}_5$, both cases (ii) and (iii) are possible, and we must compare the manufacturer's profit in these two cases to determine the equilibrium. We can obtain the optimal profits for these two cases using the w^* for the respective cases. The optimal profit for case (ii) is $\frac{(\beta-r)(r(4-3\beta)-2\beta(1-\beta))}{\beta^2(2-\beta)}$, and that for case (iii) is simply $\frac{1}{8}$. Comparing these two profits, it is easy to verify that the manufacturer would choose the first option if $r \ge \frac{\beta(12-10\beta+\sqrt{2\beta(2-\beta)})}{4(4-3\beta)} = \tilde{\rho}_2$. Since $\tilde{\rho}_2 > \tilde{\rho}_5$ holds trivially, (iii) is the equilibrium outcome only if $r \ge \tilde{\rho}_2$.

Combining the above with (A5), the optimal w and p are given by

$$w^* = \begin{cases} \frac{2(1-\beta)+r}{2(2-\beta)}, & \text{if } r < \tilde{\rho}_1 \\ \frac{2r}{\beta}-1+\frac{\beta-r}{2-\beta}, & \text{if } \tilde{\rho}_1 \le r < \tilde{\rho}_2 \\ w_0 = \frac{1}{2}, & \text{otherwise} \end{cases} \quad \text{and} \quad p^* = \begin{cases} \frac{6(1-\beta)+3r}{4(2-\beta)}, & \text{if } r < \tilde{\rho}_1 \\ \frac{r}{\beta}, & \text{if } \tilde{\rho}_1 \le r < \tilde{\rho}_2 \\ p_0 = \frac{3}{4}, & \text{otherwise} \end{cases}$$

Using these p^* and w^*, we can find the equilibrium profits for the manufacturer and retailer as π_m^* and π_r^*, respectively:

$$\pi_m^* = \begin{cases} \frac{(2(1-\beta)+r)^2}{16(2-\beta)(1-\beta)}, & \text{if } r < \tilde{\rho}_1 \\ \frac{(\beta-r)(r(4-3\beta)-2\beta(1-\beta))}{\beta^2(2-\beta)}, & \text{if } \tilde{\rho}_1 \le r < \tilde{\rho}_2 \\ \pi_{m0} = \frac{1}{8}, & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_r^* = \begin{cases} \frac{(2(1-\beta)+r)^2}{32(2-\beta)(1-\beta)}, & \text{if } r < \tilde{\rho}_1 \\ \frac{2(1-\beta)(\beta-r)^2}{\beta^2(2-\beta)}, & \text{if } \tilde{\rho}_1 \le r < \tilde{\rho}_2 \\ \pi_{r0} = \frac{1}{16}, & \text{otherwise} \end{cases}$$

We now examine to see if and when the manufacturer and the retailer are better off with piracy than without. First, since in the piracy region ($r < \tilde{\rho}_1$), the manufacturer's profit, $\pi_m^* = \frac{(2(1-\beta)+r)^2}{16(2-\beta)(1-\beta)}$, is increasing in r, equating this profit to the benchmark profit of $\pi_{m0} = \frac{1}{8}$ and solving for r, we find $r = \sqrt{4-2\beta(3-\beta)} - 2(1-\beta)$; of course, for it to be a valid root this r must abide by the restriction, $r < \tilde{\rho}_1$, which is equivalent to $\beta < \frac{16}{17}$. Next, in the threat region ($\tilde{\rho}_1 \le r < \tilde{\rho}_2$), the manufacturer's profit, $\pi_m^* = \frac{(\beta-r)(r(4-3\beta)-2\beta(1-\beta))}{\beta^2(2-\beta)}$, can never be less than π_{m0}. In other words, for all $\beta \le \frac{16}{17}$, a necessary and sufficient for the manufacturer to be better off is $\sqrt{4-2\beta(3-\beta)} - 2(1-\beta) < r < \tilde{\rho}_2$.

The case of $\beta > \frac{16}{17}$ is somewhat different. Here, the threat region takes over at a lower r; the profit function for the threat region meets the benchmark profit, $\pi_{m0} = \frac{1}{8}$, two times, first at point $\tilde{\rho}_2^c$ and then again at $\tilde{\rho}_2$, where $\tilde{\rho}_2^c$ is the root conjugate to $\tilde{\rho}_2$ and is given by

$$\tilde{\rho}_2^c = \frac{\beta\left(12 - 10\beta - \sqrt{2\beta(2-\beta)}\right)}{4(4-3\beta)}$$

Therefore, for all $\beta > \frac{16}{17}$, the manufacturer would be better off if and only if $\tilde{\rho}_2^c < r < \tilde{\rho}_2$. Defining

$$\tilde{\rho}_3 = \begin{cases} \sqrt{4-2\beta(3-\beta)}-2(1-\beta), & \text{if } \beta \le \frac{16}{17} \\ \tilde{\rho}_2^c = \frac{\beta\left(12-10\beta-\sqrt{2\beta(2-\beta)}\right)}{4(4-3\beta)}, & \text{otherwise} \end{cases}$$

it is clear that the manufacturer is better off if $\tilde{\rho}_3 < r < \tilde{\rho}_2$.

Next, we consider the retailer. The retailer's profit, $\pi_r^* = \frac{(2(1-\beta)+r)^2}{32(2-\beta)(1-\beta)}$, is also increasing in r in the piracy region ($r < \tilde{\rho}_1$). Therefore, as before, equating this profit to the benchmark profit of $\pi_{r0} = \frac{1}{16}$ and solving for r, we find that the retailer would also be better off if $r > \sqrt{4-2\beta(3-\beta)} - 2(1-\beta)$ and $\beta \le \frac{16}{17}$. In the threat region ($\tilde{\rho}_1 \le r < \tilde{\rho}_2$), the retailer's profit, $\pi_r = \frac{2(1-\beta)(\beta-r)^2}{\beta^2(2-\beta)}$, is decreasing in r. This profit is greater than or equal to $\pi_{r0} = \frac{1}{16}$ if and only if $r < \frac{\beta\left((1-\beta)\left(8-2\sqrt{\frac{2(1-\beta)}{2-\beta}}\right)-\beta\sqrt{\frac{2(1-\beta)}{2-\beta}}\right)}{8(1-\beta)}$ and $\beta \le \frac{16}{17}$. We define

$$\tilde{\rho}_4 = \begin{cases} \frac{\beta\left((1-\beta)\left(8-2\sqrt{\frac{2(1-\beta)}{2-\beta}}\right)-\beta\sqrt{\frac{2(1-\beta)}{2-\beta}}\right)}{8(1-\beta)}, & \text{if } \beta \le \frac{16}{17} \\ \tilde{\rho}_2^c = \frac{\beta\left(12-10\beta-\sqrt{2\beta(2-\beta)}\right)}{4(4-3\beta)}, & \text{otherwise} \end{cases}$$

Clearly then, the retailer is better off in the presence of piracy or its threat if $\tilde{\rho}_3 < r < \tilde{\rho}_4$.

Subscription Services and Product Bundling

Assuming that the consumers' valuation for the bundle still follows a uniform distribution over [0,1], and the same degradation factor, β, for both types of pirated content, it is easy to verify that the legal demand is still given by $q(p)$ in (1). The retailer chooses p in order to maximize its profit $\pi_r(p) = (p - w_1 - w_2)\, q(p)$.

If $q(p) = 1 - \frac{p-r}{1-\beta}$, then $\pi_r(p) = (p - w_1 - w_2)\left(1 - \frac{p-r}{1-\beta}\right)$, and by taking the derivative with respect to p, we obtain

$$\frac{\partial \pi_r}{\partial p} = 1 - \frac{2p-r-(w_1+w_2)}{1-\beta} \tag{A6}$$

Since $\frac{\partial^2 \pi_r}{\partial p^2} = -\frac{2}{1-\beta} < 0$, the first-order condition results in $p^*(w\_1, w_2) = \frac{1-\beta+r+w_1+w_2}{2}$, which, according to (1), must be greater than $\frac{r}{\beta}$, or $w_1 + w_2 > \frac{2r}{\beta} - (1 - \beta + r)$, for this solution to be valid.

If, on the other hand, $q(p) = 1 - p$, then $\pi_r(p) = (p - w_1 - w_2)(1 - p)$, so we get

$$\frac{\partial \pi_r}{\partial p} = 1 - 2p + w_1 + w_2 \tag{A7}$$

Since the second-order condition is trivially satisfied, we can set (A7) to zero to obtain $p^*(w) = \frac{1+w_1+w_2}{2}$, which must be smaller than $\frac{r}{\beta}$, or $w_1 + w_2 < \frac{2r}{\beta} - 1$, for this solution to be valid.

Now, for moderate values of $(w_1 + w_2)$, that is, if $\frac{2r}{\beta} - 1 \leq w_1 + w_2 \leq \frac{2r}{\beta} - (1 - \beta + r)$, $\frac{\partial \pi_r}{\partial p}$ in (A6) is negative, whereas that in (A7) is positive. Naturally, the optimal p is simply $\frac{r}{\beta}$. Taken together, the optimal retail price, $p^*(w_1, w_2)$, can be expressed as

$$p^*(w_1, w_2) = \begin{cases} \frac{1-\beta+r+w_1+w_2}{2}, & \text{if } w_1 + w_2 > \frac{2r}{\beta} - (1 - \beta + r) \\ \frac{r}{\beta}, & \text{if } \frac{2r}{\beta} - 1 \leq w_1 + w_2 \leq \frac{2r}{\beta} - (1 - \beta + r) \\ \frac{1+w_1+w_2}{2}, & \text{otherwise} \end{cases} \tag{A8}$$

Now consider the move from manufacturer 1. It anticipates this reaction from the retailer and, given the other manufacturer's wholesale price, w_2, sets its own optimal wholesale price $w_1(w_2)$ to maximize $\pi_{m_1}(w_1, w_2) = w_1\, q(p^*(w_1, w_2))$. As before, we have three cases to consider: (i) $w_1 + w_2 > \frac{2r}{\beta} - (1 - \beta + r)$, (ii) $\frac{2r}{\beta} - 1 \leq w_1 + w_2 \leq \frac{2r}{\beta} - (1 - \beta + r)$, and (iii) $w_1 + w_2 < \frac{2r}{\beta} - 1$. For case (i), manufacturer 1 gets a profit of

$$\pi_{m_1} = \frac{w_1(1 - \beta + r - (w_1 + w_2))}{2(1 - \beta)}$$

Since $\frac{\partial^2 \pi_{m_1}}{\partial w_1^2} = -\frac{1}{1-\beta} < 0$, solving the first order condition, $\frac{\partial \pi_{m_1}}{\partial w_1} = \frac{1-\beta+r-2w_1-w_2}{2\,(1-\beta)} = 0$, we get the optimal response function: $w_1^1(w_2) = \frac{1-\beta+r-w_2}{2}$. Similar logic applied to manufacturer 2 gives us its response function as: $w_2^1(w_1) = \frac{1-\beta+r-w_1}{2}$. Simultaneously solving the two response functions, we obtain $w_1^{1*} = w_2^{1*} = \frac{1-\beta+r}{3}$. For this equilibrium to be valid, $(w_1^{1*} + w_2^{1*})$ must be greater than $\frac{2r}{\beta} - (1 - \beta + r)$, which is equivalent to $r \leq \frac{5\beta(1-\beta)}{6-5\beta} = \tilde{\rho}_1$.

For case (ii), $p^* = \frac{r}{\beta}$. The manufacturers, unwilling to leave money on the table, always choose the highest value from the range $\frac{2r}{\beta} - 1 \leq w_1 + w_2 \leq \frac{2r}{\beta} - (1 - \beta + r)$, resulting in response functions: $w_1^2(w_2) = \frac{2r}{\beta} - (1-\beta+r) - w_2$ and $w_2^2(w_1) = \frac{2r}{\beta} - (1-\beta+r) - w_1$. Once again, simultaneously solving the two response functions, we get $w_1^{2*} = w_2^{2*} = \frac{r}{\beta} - \frac{1-\beta+r}{2}$. To determine the validity of this solution, we note that it must be incentive compatible in the sense that a manufacturer must not have the incentive to deviate to case (i) if the other manufacturer is in case (ii). However, it turns out that

$$\pi_{m_1}\big|_{w_1=w_1^1(w_2^{2*}), w_2=w_2^{2*}} - \pi_{m_1}\big|_{w_1=w_1^2(w_2^{2*}), w_2=w_2^{2*}} = \frac{\left(5\beta(1-\beta) - r(6-5\beta)\right)^2}{32\beta^2(1-\beta)} \geq 0$$

This is expected; after all, the interior response for a manufacturer should always be better than the boundary response, meaning that case (i) dominates case (ii). However, as we have shown above, case (i) is a valid equilibrium only if $r < \tilde{\rho}_1$. Therefore, for all $r < \tilde{\rho}_1$, the manufacturer would have an incentive to switch from case (ii) to case (i), so case (ii) cannot be a valid equilibrium there. In contrast, if $r \geq \tilde{\rho}_1$, case (i) is not valid, so case (ii) can be a valid equilibrium there.

Finally, in case (iii), $p^*(w_1, w_2) = \frac{1+w_1+w_2}{2}$, and manufacturer 1 gets a profit of $\pi_{m_1} = \frac{w_1(1-(w_1+w_2))}{2}$, which is convex in w_1 and can be easily maximized using the first order condition. The resulting response functions are $w_1^3(w_2) = \frac{1-w_2}{2}$ and $w_2^3(w_1) = \frac{1-w_1}{2}$, implying $w_1^{3*} = w_2^{3*} =$

$\frac{1}{3}$. For this solution to be valid, we must have $w_1^{3*} + w_2^{3*} < \frac{2r}{\beta} - 1$, that is, $r \geq \frac{5\beta}{6} = \tilde{\rho}_5$. Comparing manufacturers' profits in cases (ii) and (iii), we find that case (ii) with prevail over case (iii) if $r < \tilde{\rho}_2 = \frac{\beta(9-6\beta+\sqrt{1+4\beta})}{6(2-\beta)}$. It is easy to verify that $\tilde{\rho}_2 > \tilde{\rho}_5$, making the overall solution spanning the three cases complete.

With the closed-form solution for w_1^* and w_2^*, we can derive p^* from (A8). Therefore, the equilibrium solution is given by

$$w_1^* = w_2^* = \begin{cases} \frac{1-\beta+r}{3}, & \text{if } r < \tilde{\rho}_1 \\ \frac{r}{\beta} - \frac{1-\beta+r}{2}, & \text{if } \tilde{\rho}_1 \leq r < \tilde{\rho}_2, \\ w_0 = \frac{1}{3}, & \text{otherwise} \end{cases} \quad \text{and} \quad p^* = \begin{cases} \frac{5(1-\beta+r)}{6}, & \text{if } r < \tilde{\rho}_1 \\ \frac{r}{\beta}, & \text{if } \tilde{\rho}_1 \leq r < \tilde{\rho}_2 \\ p_0 = \frac{5}{6}, & \text{otherwise} \end{cases}$$

where, as stated earlier, $\tilde{\rho}_1 = \frac{5\beta(1-\beta)}{6-5\beta}$ and $\tilde{\rho}_2 = \frac{\beta(9-6\beta+\sqrt{1+4\beta})}{6(2-\beta)}$. From these, we can now obtain the profits for the manufacturers and the retailer as

$$\pi_{m_1}^* = \pi_{m_2}^* = \begin{cases} \frac{(1-\beta+r)^2}{18(1-\beta)}, & \text{if } r < \tilde{\rho}_1 \\ \frac{(\beta-r)(r-(1-\beta)(\beta-r))}{2\beta^2}, & \text{if } \tilde{\rho}_1 \leq r < \tilde{\rho}_2 \\ \pi_{m0} = \frac{1}{18}, & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_r^* = \begin{cases} \frac{(1-\beta+r)^2}{36(1-\beta)}, & \text{if } r < \tilde{\rho}_1 \\ \frac{(1-\beta)(\beta-r)^2}{\beta^2}, & \text{if } \tilde{\rho}_1 \leq r < \tilde{\rho}_2 \\ \pi_{r0} = \frac{1}{36}, & \text{otherwise} \end{cases}$$

When these profits are compared to their benchmark values, we can obtain the win-win window similar to the one in Theorem 1. First, since in the piracy region ($r < \tilde{\rho}_1$), the manufacturers' profits, $\pi_{m_1}^* = \pi_{m_2}^* = \frac{(1-\beta+r)^2}{18(1-\beta)}$, are increasing in r, equating these profits to the benchmark profits of $\pi_{m0} = \frac{1}{18}$ and solving for r, we find $r = \sqrt{1-\beta} - (1-\beta)$; of course, for it to be a valid root this r must abide by the restriction $r < \tilde{\rho}_1$, which is equivalent to $\beta < \frac{24}{25}$. Next, in the threat region ($\tilde{\rho}_1 \leq r < \tilde{\rho}_2$), the manufacturers' profits, $\pi_{m_1}^* = \pi_{m_2}^* = \frac{(\beta-r)(r-(1-\beta)(\beta-r))}{2\beta^2}$, can never be less than π_{m0}. In other words, for all $\beta \leq \frac{24}{25}$, a necessary and sufficient condition for the manufacturer to be better off is $\sqrt{1-\beta} - (1-\beta) < r < \tilde{\rho}_2$.

The case of $\beta > \frac{24}{25}$ is somewhat different. Here, the threat region takes over at a lower r; the profit function for the threat region meets the benchmark profit, $\pi_{m0} = \frac{1}{18}$, two times, first at point $\tilde{\rho}_2^c$ and then again at $\tilde{\rho}_2$, where $\tilde{\rho}_2^c$ is the root conjugate to $\tilde{\rho}_2$ and is given by

$$\tilde{\rho}_2^c = \frac{\beta\left(9 - 6\beta - \sqrt{1+4\beta}\right)}{6(2-\beta)}$$

Therefore, for all $\beta > \frac{24}{25}$, the manufacturer would be better off if and only if $\tilde{\rho}_2^c < r < \tilde{\rho}_2$. Defining

$$\tilde{\rho}_3 = \begin{cases} \sqrt{1-\beta} - (1-\beta), & \text{if } \beta \leq \frac{24}{25} \\ \tilde{\rho}_2^c = \frac{\beta(9-6\beta-\sqrt{1+4\beta})}{6(2-\beta)}, & \text{otherwise} \end{cases}$$

it is clear that the manufacturers are better off if $\tilde{\rho}_3 < r < \tilde{\rho}_2$.

Next, we consider the retailer. The retailer's profit, $\pi_{r^*} = \frac{(1-\beta+r)^2}{36(1-\beta)}$, is also increasing in r in the piracy region ($r < \tilde{\rho}_1$). Therefore, as before, equating this profit to the benchmark profit of $\pi_{r0} = \frac{1}{36}$ and solving for r, we find that the retailer would also be better off if $r > \sqrt{1-\beta} - (1-\beta)$ and $\beta \leq \frac{24}{25}$. In the threat region ($\tilde{\rho}_1 \leq r < \tilde{\rho}_2$), the retailer's profit, $\pi_r^* = \frac{(1-\beta)(\beta-r)^2}{\beta^2}$, is decreasing in r. This profit is greater than or equal to $\pi_{r0} = \frac{1}{36}$ if and only if $r < \beta\left(1 - \frac{1}{6\sqrt{1-\beta}}\right)$ and $\beta \leq \frac{24}{25}$. We define

$$\tilde{\rho}_4 = \begin{cases} \beta\left(1 - \frac{1}{6\sqrt{1-\beta}}\right), & \text{if } \beta \leq \frac{24}{25} \\ \tilde{\rho}_2^c = \frac{\beta(9-6\beta-\sqrt{1+4\beta})}{6(2-\beta)}, & \text{otherwise} \end{cases}$$

Clearly then, the retailer is better off in the presence of piracy or its threat if $\tilde{\rho}_3 < r < \tilde{\rho}_4$.

Piracy Cost Recouped by the Legal Channel

Recall that the demands for the legal and illegal versions at a given retail price p are exactly as those in our original model in (1). However, in this extension, the manufacturer and retailer also make an additional $\mu\lambda r$ and $(1-\mu)\lambda r$, respectively, for every unit of illegal product sold. Using (1), the resulting profit functions for the manufacturer and the retailer can then be written as

$$\pi_m = \begin{cases} w\left(1 - \frac{p-r}{1-\beta}\right) + \mu\lambda r\left(\frac{p-r}{1-\beta} - \frac{r}{\beta}\right), & \text{if } p > \frac{r}{\beta} \\ w(1-p), & \text{otherwise} \end{cases} \tag{A9}$$

$$\pi_r = \begin{cases} (p-w)\left(1 - \frac{p-r}{1-\beta}\right) + (1-\mu)\lambda r\left(\frac{p-r}{1-\beta} - \frac{r}{\beta}\right), & \text{if } p > \frac{r}{\beta} \\ (p-w)(1-p), & \text{otherwise} \end{cases} \tag{A10}$$

As a result, the optimal prices differ from those in the original model. The optimal retail price for a given w, $p^*(w)$, can now be found by maximizing π_r in (A10). Repeating exactly the same method we used for deriving Lemma 1, we can easily derive an analogous expression for the optimal p in this extended setup

$$p^*(w) = \begin{cases} \frac{1-\beta+r(1+\lambda(1-\mu))+w}{2}, & \text{if } w > \frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta) \\ \frac{r}{\beta}, & \text{if } \frac{2r}{\beta} - 1 \le w \le \frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta) \\ \frac{1+w}{2}, & \text{otherwise} \end{cases}$$

Note that, when $\lambda = 0$, this $p^*(w)$ coincides with that given in Lemma 1. We are now ready to characterize the new ρ_i thresholds, $i \in \{1,2,3,4,5\}$; to avoid confusion with our original notation, we denote the new ones as $\tilde{\rho}_i$ here.

Once the retailer's response, $p^*(w)$, is known, the manufacturer's problem is to maximize $w(1-p^*(w))$. It is clear from the expression of $p^*(w)$ above that we have three cases to consider: (i) $w > \frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta)$, (ii) $\frac{2r}{\beta} - 1 \le w \le \frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta)$, and (iii) $w < \frac{2r}{\beta} - 1$.

For case (i), $p^*(w) = \frac{1-\beta+r(1+\lambda(1-\mu))+w}{2}$, and the first order condition with respect to w results in $w^* = \frac{1-\beta+r(1+\lambda(2\mu-1))}{2}$. This solution must be greater than $\frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta)$, which leads to $r < \tilde{\rho}_1 = \frac{3\beta(1-\beta)}{4-3\beta(1+\lambda)}$.

For case (ii), $p^* = \frac{r}{\beta}$. The manufacturer, unwilling to leave money on the table, always chooses the highest value from the range $\left[\frac{2r}{\beta} - 1, \frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta)\right]$, resulting in $w^* = \frac{2r}{\beta} - r(1+\lambda(1-\mu)) - (1-\beta)$. This equilibrium is valid across all $r \le \beta$. As was the case in our original model, $r > \beta$ still falls under case (iii), in which no consumer considers the pirated product as an option.

Finally, in case (iii), $p^*(w) = \frac{1+w}{2}$, which leads to $w^* = \frac{1}{2}$. This w^* must be less than $\frac{2r}{\beta} - 1$, implying that $r \ge \frac{3\beta}{4} = \tilde{\rho}_5$ must hold for case (iii) to occur.

Now, case (ii) is the only valid equilibrium if $\tilde{\rho}_1 \le r \le \tilde{\rho}_5$. On one hand, if $r < \tilde{\rho}_1$, both cases (i) and (ii) are valid. However, the optimal profit from an interior solution ought to be higher, which immediately implies that case (i) is the equilibrium outcome for $r < \tilde{\rho}_1$. If, on the other hand, $r > \tilde{\rho}_5$, both cases (ii) and (iii) are possible, and we must compare the manufacturer's profit in these two cases to determine the equilibrium. We can obtain the optimal profits for these two cases using the w^* for the respective cases. The optimal profit for case (ii) is $\frac{(\beta-r)(r(2-\beta(1+\lambda(1-\mu)))-\beta(1-\beta))}{\beta^2}$. The optimal profit for case (iii) is simply $\frac{1}{8}$. Accordingly, $\tilde{\rho}_2$, the boundary between the limit and benchmark regions, is given by

$$\tilde{\rho}_2 = \beta\left(1 - \frac{2(1-\beta\lambda(1-\mu)) - \sqrt{2\beta(1-\lambda(1-\mu)(3-2\beta\lambda(1-\mu)))}}{4\left(2-\beta\left(1+\lambda(1-\mu)\right)\right)}\right)$$

Now, let us turn to the win-win region. Unlike in our original model, it is no longer true that the manufacturer wins whenever the retailer wins. This is because, when μ is small and consequently $(1-\mu)$ is large, the retailer may win while the manufacturer loses. Therefore, to find $\tilde{\rho}_3$, we must first find the thresholds for the manufacturer and retailer separately. Once we know the thresholds above which the manufacturer and retailer are better off, we can take their maximum to determine $\tilde{\rho}_3$. The threshold for the manufacturer, $\tilde{\rho}_{3m}$, is obtained by equating the manufacturer's profit in the piracy region with that in the benchmark region. The manufacturer's profit in the piracy region is

$$\frac{2\beta r(1-\beta)(1-\lambda(1-4\mu))+(1-\beta)^2+\ r^2(\beta(1-\lambda)^2-8\lambda\mu(1-\beta))}{8\beta(1-\beta)}$$

Since the profit in the benchmark region is $\frac{1}{8}$, we get

$$\tilde{\rho}_{3m}=\frac{\beta}{1-\lambda(1-4\mu)+\sqrt{\frac{1-\lambda(2-\lambda(1-8\mu(1-\beta)(1-2\mu)))}{1-\beta}}}$$

Similarly, we can solve the retailer's threshold. Its profit in the piracy region is

$$\frac{2\beta r(1-\beta)(1+\lambda(7-8\mu))+\beta(1-\beta)^2+\ r^2(\beta(1-\lambda)^2-16\lambda(1-\beta)(1-\mu))}{16\beta(1-\beta)}$$

The retailer makes $\frac{1}{16}$ in the benchmark region. It immediately follows that

$$\tilde{\rho}_{3r}=\frac{\beta}{1-\lambda(7-8\mu)+\sqrt{\frac{1-\lambda(2-\lambda((7-8\mu)^2-16\beta(1-\mu)(3-4\mu)))}{1-\beta}}}$$

It is easy to verify that $\tilde{\rho}_{3m}>\tilde{\rho}_{3r}$ for $\mu<\frac{2}{3}$, which leads to

$$\tilde{\rho}_3=\begin{cases}\tilde{\rho}_{3m}, & \text{if } \mu<\frac{2}{3}\\ \tilde{\rho}_{3r}, & \text{otherwise}\end{cases}$$

Now, to solve for $\tilde{\rho}_4$, we need to compare the profit in case (ii) with the benchmark profit in case (iii). We again do this exercise separately for the manufacturer and retailer to obtain $\tilde{\rho}_{4m}$ and $\tilde{\rho}_{4r}$, respectively. The upper bound of the win-win region, $\tilde{\rho}_4$, is then the smaller of these two thresholds. Note that, by definition,

$$\tilde{\rho}_{4m}=\tilde{\rho}_2$$

and $\tilde{\rho}_{4r}$ is the solution of

$$\frac{(\beta-r)(\beta(1-\beta)-r(1-\beta(1+\lambda(1-\mu))))}{\beta^2}=\frac{1}{16}$$

Therefore,

$$\tilde{\rho}_{4r}=\beta\left(1+\frac{2\beta\lambda(1-\mu)-\sqrt{1-\beta(1+\lambda(1-\mu)(1-4\beta\lambda(1-\mu)))}}{4(1-\beta(1+\lambda(1-\mu)))}\right)$$

Comparing $\tilde{\rho}_{4m}$ with $\tilde{\rho}_{4r}$, we can derive $\tilde{\rho}_4$:

$$\tilde{\rho}_4=\begin{cases}\tilde{\rho}_{4m}, & \text{if } \lambda(1-\mu)>\frac{1}{3}\\ \tilde{\rho}_{4r}, & \text{otherwise}\end{cases}$$

Finally, as shown in the paper, for the win-win region to exist, both $\tilde{\rho}_3$ and $\tilde{\rho}_4$ must be real and must satisfy $\tilde{\rho}_3<\tilde{\rho}_4$.

Network Effect

Since we now assume a consumer's valuation to $v(1+\Gamma)$, the demand for the legal product becomes

$$q(p) = \begin{cases} 1 - \frac{p-r}{(1-\beta)(1+\Gamma)}, & \text{if } p > \frac{r}{\beta} \\ 1 - \frac{p}{1+\Gamma}, & \text{otherwise} \end{cases}$$

which can also be rewritten as

$$q(p) = \begin{cases} 1 - \frac{p-r}{(1-\beta)(1+\Gamma_{\text{piracy}})}, & \text{if } p > \frac{r}{\beta} \\ 1 - \frac{p}{1+\Gamma_{\text{threat}}}, & \text{if } p = \frac{r}{\beta} \\ 1 - \frac{p}{1+\Gamma_{\text{benchmark}}}, & \text{otherwise} \end{cases} \quad \text{(A11)}$$

Let us first consider the piracy region $\left(p > \frac{r}{\beta}\right)$ and the threat region $\left(p = \frac{r}{\beta}\right)$, where the marginal consumer, $\bar{v}$, can be characterized by $\bar{v} = \frac{r}{\beta(1+\Gamma)}$. Since $\Gamma = \gamma(1-\bar{v})$ by definition, in a fulfilled expectations equilibrium, the following must hold:

$$\Gamma = \gamma(1-\bar{v}) = \gamma\left(1 - \frac{r}{\beta(1+\Gamma)}\right)$$

Solving this, we obtain the equilibrium Γ as follows:

$$\Gamma_{\text{piracy}} = \Gamma_{\text{threat}} = \frac{1}{2}\left(\gamma - 1 + \sqrt{(\gamma+1)^2 - \frac{4r\gamma}{\beta}}\right)$$

Now, let us consider the benchmark region where $p < \frac{r}{\beta}$. Starting with the demand expression in (A11), it is straightforward to show that the equilibrium price set by the retailer is simply $p = \frac{3(1+\Gamma)}{4}$, which means, exactly as in our original model, only a quarter of the market gets covered in equilibrium regardless of the actual value of Γ, implying that $\bar{v} = \frac{3}{4}$. Hence, the equilibrium Γ must be

$$\Gamma_{\text{benchmark}} = \gamma(1-\bar{v}) = \frac{\gamma}{4}$$

With the demand so characterized, we can now proceed to solve for the thresholds $\tilde{\rho}_i$ that are analogous to the thresholds ρ_i in Theorem 1, for $i \in \{1,2,3,4,5\}$. Recall that $\Gamma_{\text{piracy}} = \Gamma_{\text{threat}}$; we will henceforth call them both Γ_a for convenience. Likewise, we will use a shorter notation Γ_b to denote $\Gamma_{\text{benchmark}}$. We proceed exactly the same way we solved our original model. Repeating the steps in Lemma 1, we can easily derive the optimal p as

$$p^*(w) = \begin{cases} \frac{(1-\beta)(1+\Gamma_a)+r+w}{2}, & \text{if } w > \frac{2r}{\beta} - r - (1-\beta)(1+\Gamma_a) \\ \frac{r}{\beta}, & \text{if } \frac{2r}{\beta} - 1 - \Gamma_b \le w \le \frac{2r}{\beta} - r - (1-\beta)(1+\Gamma_a) \\ \frac{1+w+\Gamma_b}{2}, & \text{otherwise} \end{cases}$$

Clearly, $\gamma = 0$ implies that $\Gamma_a = \Gamma_b = 0$. As a result, when $\gamma = 0$, $p^*(w)$ above coincides with that given in Lemma 1.

Once the retailer's response $p^*(w)$ is known, the manufacturer's problem is simply to maximize $w(1-p^*(w))$. It is clear from the expression of $p^*(w)$ above that we have three cases to consider: (i) $w > \frac{2r}{\beta} - r - (1-\beta)(1+\Gamma_a)$, (ii) $\frac{2r}{\beta} - 1 - \Gamma_b \le w \le \frac{2r}{\beta} - r - (1-\beta)(1+\Gamma_a)$, and (iii) $w < \frac{2r}{\beta} - 1 - \Gamma_b$.

For case (i), $p^*(w) = \frac{(1-\beta)(1+\Gamma_a)+r+w}{2}$, and the first order condition with respect to w results in $w^* = \frac{(1-\beta)(1+\Gamma_a)+r}{2}$. This solution must be greater than $\frac{2r}{\beta} - r - (1-\beta)(1+\Gamma_a)$, which leads to $r < \tilde{\rho}_1$ where $\tilde{\rho}_1$ is the solution of

$$\frac{(1-\beta)(1+\Gamma_a)+r}{2}=\frac{2r}{\beta}-r-(1-\beta)(1+\Gamma_a)$$

Substituting $\Gamma_a=\frac{1}{2}\left(\gamma-1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)$ and subsequently solving the above, we get

$$\tilde{\rho}_1=\frac{3\beta(1-\beta)(4-3\beta+\gamma)}{(4-3\beta)^2}$$

For case(ii), $p^*=\frac{r}{\beta}$ and, once again, the manufacturer, unwilling to leave money on the table, chooses the highest value from the range $\left[\frac{2r}{\beta}-1-\Gamma_b,\frac{2r}{\beta}-r-(1-\beta)(1+\Gamma_a)\right]$, resulting in $w^*=\frac{2r}{\beta}-r-(1-\beta)(1+\Gamma_a)$. As was the case in our original model, this equilibrium is valid across all $r\leq\beta$ and the case of $r>\beta$ falls under case (iii) where the pirated product is not an option.

Finally, in case (iii), $p^*(w)=\frac{1+w+\Gamma_b}{2}$, which leads to $w^*=\frac{1+\Gamma_b}{2}$. This w^*must be less than $\frac{2r}{\beta}-1-\Gamma_b$, where $\Gamma_b=\frac{\gamma}{4}$, implying that $r\geq\frac{3\beta}{4}\left(1+\frac{\gamma}{4}\right)=\tilde{\rho}_5$ must hold for case (iii) to occur.

Now, case (ii) is the only valid equilibrium if $\tilde{\rho}_1\leq r\leq\tilde{\rho}_5$. On the one hand, if $r<\tilde{\rho}_1$, both cases (i) and (ii) are valid. However, the optimal profit from an interior solution ought to be higher, which immediately implies that case (i) is the equilibrium outcome for $r<\tilde{\rho}_1$. If, on the other hand, $r>\tilde{\rho}_5$, both cases (ii) and (iii) are possible, and we must compare the manufacturer's profit in these two cases to determine the equilibrium. We can obtain the optimal profits for these two cases using the w^* for the respective cases. The optimal profit for case (ii) is $\left(\frac{2r}{\beta}-r-(1-\beta)(1+\Gamma_a)\right)\left(1-\frac{r}{\beta(1+\Gamma_a)}\right)$, where, as before, $\Gamma_a=\frac{1}{2}\left(\gamma-1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)$. The optimal profit for case (iii) is $\frac{1+\Gamma_b}{8}$, where $\Gamma_b=\frac{\gamma}{4}$. Hence, $\tilde{\rho}_2$ can be obtained as the larger of the two positive roots of

$$\left(\frac{2r}{\beta}-r-\frac{1-\beta}{2}\left(\gamma+1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)\right)\left(1-\frac{r}{\frac{\beta}{2}\left(\gamma+1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)}\right)=\frac{1+\frac{\gamma}{4}}{8}$$

A closed-form solution does exist, but the size of the expression precludes it from this appendix.

Now, let us turn to the win-win region. To solve for $\tilde{\rho}_3$, we need to equate the retailer's profit in case (i) with the benchmark profit, that is, the profit in case (iii). The profit in case (i) is $\frac{(r+(1-\beta)(1+\Gamma_a))^2}{16\,(1-\beta)(1+\Gamma_a)}$ while that in case (iii) is $\frac{1+\Gamma_b}{16}$. Hence, $\tilde{\rho}_3$ is the root in $[0,\tilde{\rho}_1]$ of the following:

$$\frac{\left(r+\frac{1-\beta}{2}\left(\gamma+1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)\right)^2}{8\,(1-\beta)\left(\gamma+1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)}=\frac{1+\frac{\gamma}{4}}{16}$$

Again, although a closed form solution exists, it is simply too long and cumbersome to report here.

Finally, to obtain $\tilde{\rho}_4$, we need to equate the retailer's profit in case (ii) with the benchmark profit. The profit in case (ii) is $\frac{(1-\beta)(\beta(1+\Gamma_a)-r)^2}{\beta^2(1+\Gamma_a)}$ and that in case (iii) is as mentioned above. Thus, $\tilde{\rho}_4$ is the root in $[\tilde{\rho}_1,\tilde{\rho}_2]$ of the following:

$$\frac{(1-\beta)\left(\frac{\beta}{2}\left(\gamma+1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)-r\right)^2}{\frac{\beta^2}{2}\left(\gamma+1+\sqrt{(\gamma+1)^2-\frac{4r\gamma}{\beta}}\right)}=\frac{1+\frac{\gamma}{4}}{16}$$

When $\gamma\rightarrow 0$, all the $\tilde{\rho}_i$ thresholds, $i\in\{1,2,3,4,5\}$, nicely converge to ρ_i in our original model.

Downstream Competition

Given the straightforward setting of a horizontal market, the fraction of consumers who prefers retailer A B is simply $\left(\frac{1}{2}+\frac{p_B-p_A}{2\delta}\right)$. Likewise, the remaining $\left(\frac{1}{2}+\frac{p_A-p_B}{2\delta}\right)$ fraction prefers B to A. It is easy to see that, when δ becomes large, the two markets separate; essentially, the weakened competition empowers the individual retailers as local monopolies, and our earlier results apply.

Now, irrespective of the value of δ, each consumer has to make a choice among: (i) buying the legal product from his preferred retailer, (ii) using an illegal copy, and (iii) not using the product at all. We consider this choice to be independent of the consumer's preference for a retailer. In other words, we continue to assume that this choice is still governed by the IR and IC constraints discussed in the consumer behavior section in the paper. Accordingly, the legal demand for retailer A can now be expressed as

$$q_A(p_A,p_B)=\begin{cases}\left(\frac{1}{2}+\frac{p_B-p_A}{2\delta}\right)\left(1-\frac{p_A-r}{1-\beta}\right), & \text{if } p_A>\frac{r}{\beta}\\ \left(\frac{1}{2}+\frac{p_B-p_A}{2\delta}\right)(1-p_A), & \text{otherwise}\end{cases} \tag{A12}$$

Retailer A maximizes $(p_A-w)q_A(p_A,p_B)$, and retailer B, $(p_B-w)q_B(p_B,p_A)$. As before, three regions emerge and a retailer prefers to employ the limit price only when w is moderate. Specifically, retailer A chooses $p_A=\frac{r}{\beta}$ if $w_{AL}\le w\le w_{AH}$, where

$$w_{AL}=\frac{3r^2+\beta^2(p_B+\delta)-2\beta r(1+p_B+\delta)}{\beta(2r-\beta(1+p_B+\delta))} \quad \text{and}$$
$$w_{AH}=\frac{r^2(3-2\beta)+\beta^2(1-\beta)(p_B+\delta)-\beta r(p_B(2-\beta)+2(1+\delta)-\beta(2+\delta))}{\beta(r(2-\beta)-\beta(1+p_B-\beta+\delta))}$$

A similar range exists for retailer B as well. Therefore, a symmetric equilibrium with $p_A=p_B=\frac{r}{\beta}$ is possible only if w is between the following two limits:

$$w_L=w_{AL}|_{p_B=\frac{r}{\beta}}=\frac{\beta r(1+2\delta)-r^2-\beta^2\delta}{\beta(\beta(1+\delta)-r)} \quad \text{and}$$
$$w_H=w_{AH}|_{p_B=\frac{r}{\beta}}=\frac{r(1-\beta)(\beta-r)+\beta\delta(r(2-\beta)-\beta(1-\beta))}{\beta(\beta(1+\delta)-r-\beta(\beta-r))}$$

Note that, if the manufacturer sets $w>w_H$, the only possible symmetric equilibrium is the one in which both retailers name a price above $\frac{r}{\beta}$. Retailer A's optimal price in this equilibrium is obtained from

$$\frac{\partial}{\partial p_A}\left((p_A-w)\left(1-\frac{p_A-r}{1-\beta}\right)\left(\frac{1}{2}+\frac{p_B-p_A}{2\delta}\right)\right)\Bigg|_{p_B=p_A}=0$$

leading to $p_A=p_B=\frac{1-\beta+r+w+2\delta-\sqrt{(1-\beta+r-w)^2+4\delta^2}}{2}$. On the other hand, if the manufacturer chooses a wholesales price below w_L, the symmetric equilibrium of interest would be the one in which $p_A=p_B<\frac{r}{\beta}$. Retailer A's first order condition in this case is

$$\frac{\partial}{\partial p_A}\left((p_A-w)(1-p_A)\left(\frac{1}{2}+\frac{p_B-p_A}{2\delta}\right)\right)\Bigg|_{p_B=p_A}=0$$

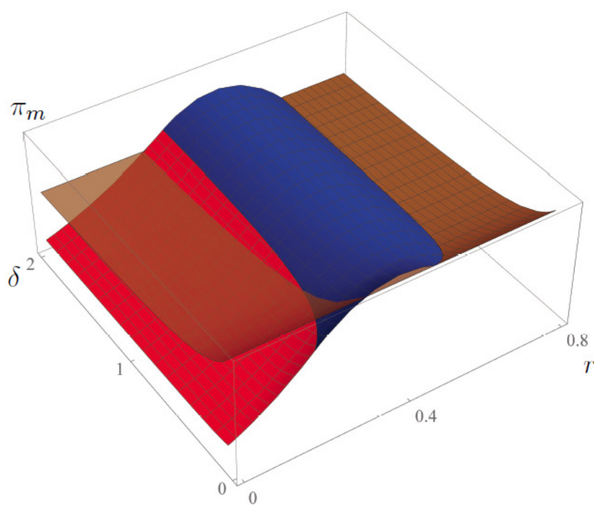
which leads to $p_A=p_B=\frac{1+w+2\delta-\sqrt{(1-w)^2+4\delta^2}}{2}$.

Putting all of the above elements together, in a symmetric equilibrium, the optimal retail price for a given w, $p^*(w)=p_A^*(w)=p_B^*(w)$, must satisfy

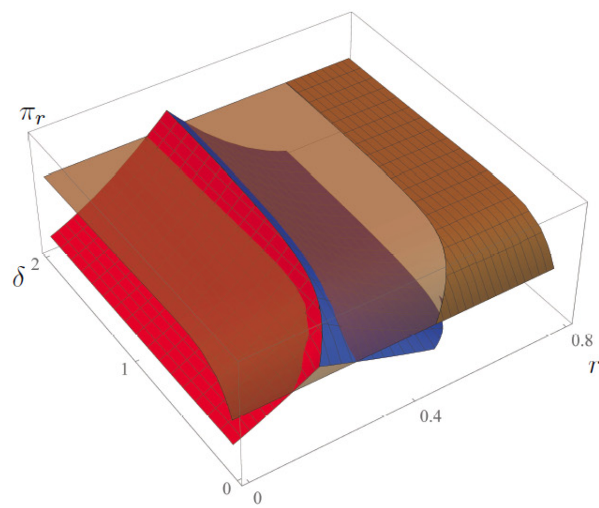
$$p^*(w)=\begin{cases}\frac{1-\beta+r+w+2\delta-\sqrt{(1-\beta+r-w)^2+4\delta^2}}{2}, & \text{if } w>w_H\\ \frac{r}{\beta}, & \text{if } w_L\le w\le w_H\\ \frac{1+w+2\delta-\sqrt{(1-w)^2+4\delta^2}}{2}, & \text{otherwise}\end{cases}$$

Since this expression is similar to the one in Lemma 1 in the paper, the rest of the derivation of the equilibrium is not conceptually any harder. In particular, given this $p^*(w)$, the manufacturer chooses w^*, the optimal w that maximizes its profit, $\pi_m(w) = 2w \times (q^*(p^*(w))$, where $q^*(p^*(w))$ is obtained by setting $p_A = p_B = p^*(w)$ in~(A12). In the piracy region ($w > w_H$), as well as in the benchmark region ($w < w_L$), the manufacturer's profit is concave in the region of interest, and a unique w^* can be found from the first order condition, although the size of its expression in Mathematica precludes reporting it in this appendix. Finally, in the threat region ($w_L \leq w \leq w_H$), the manufacturer prefers w_H to any other $w \in [w_L, w_H]$ while inducing the retailers to choose $p^*(w) = \frac{r}{\beta}$ in equilibrium.

By a chain of backward substitutions of this w^*, we can find the optimal retail price, $p^*(w^*)$ and the optimal demand $q^*(p^*(w^*))$. Therefore, the equilibrium profits of the manufacturer and retailers can also be found. Fortunately, unique closed form expressions still exist; it is just that they are simply too large to report here. Instead, we illustrate their behavior in Figure A4, where these profits are plotted as functions r and β. Once again, even in this case, the red-blue humps over the benchmark level, reminiscent of a win-win window, are unmistakably visible. Therefore, to establish the existence of a win-win window, all that remains is to show that there is some overlap between the two humps in the two profit plots. In particular, let $(\tilde{\rho}_{3m}, \tilde{\rho}_2)$ be the manufacturer's winning window and $(\tilde{\rho}_{3r}, \tilde{\rho}_4)$, the retailers'. These windows can be analytically obtained and plotted, as shown in Figure 12. The overlap between them is clearly visible in the figure.



(a) Manufacturer's Proft



(b) Retailer's Profit

Figure A4. Profit as a Function of r and δ; $\beta = 0.75$

Appendix B

Proofs

Proof of Lemma 1

If $q(p) = 1 - \frac{p-r}{1-\beta}$, then $\pi_r(p) = (p - w)\left(1 - \frac{p-r}{1-\beta}\right)$, implying

$$\frac{\partial \pi_r}{\partial p} = 1 - \frac{2p-r-w}{1-\beta} \tag{B1}$$

Since $\frac{\partial^2 \pi_r}{\partial p^2} = -\frac{2}{1-\beta} < 0$, the first order condition results in $p^*(w) = \frac{1}{2}(1 - \beta + r + w)$, which according to (1), must be greater than $\frac{r}{\beta}$, or $w > \frac{2r}{\beta} - (1 - \beta + r)$, for this solution to be valid.

If, on the other hand, $q(p) = 1 - p$, then $\pi_r(p) = (p - w)(1 - p)$, resulting in

$$\frac{\partial \pi_r}{\partial p} = 1 - 2p + w \tag{B2}$$

Furthermore, since $\frac{\partial^2 \pi_r}{\partial p^2} = -2 < 0$, $\frac{\partial \pi_r}{\partial p} = 0$ results in $p^*(w) = \frac{1+w}{2}$, which must be smaller than $\frac{r}{\beta}$, or $w < \frac{2r}{\beta} - 1$, for this solution to be valid.

Now, for moderate values of w, that is, if $\frac{2r}{\beta} - 1 \leq w \leq \frac{2r}{\beta} - (1 - \beta + r)$, $\frac{\partial \pi_r}{\partial p}$ given by (B1) is negative whereas that given by (B2) is positive. Naturally, the optimal p is simply $\frac{r}{\beta}$. ■

Proof of Proposition 1

From Lemma 1, it is evident that we have three cases to consider: (i) w $> \frac{2r}{\beta} - (1 - \beta + r)$, (ii) $\frac{2r}{\beta} - 1 \leq w \leq \frac{2r}{\beta} - (1 - \beta + r)$ and (iii) $w < \frac{2r}{\beta} - 1$.

For case (i), we substitute $p^*(w) = \frac{1}{2}(1 - \beta + r + w)$ into (1) to obtain the manufacturer's profit

$$\pi_m = \frac{w(1-\beta+r-w)}{2(1-\beta)} \tag{B3}$$

Since $\frac{\partial^2 \pi_m}{\partial w^2} = -\frac{1}{1-\beta} < 0$, the first order condition, $\frac{\partial \pi_m}{\partial m} = \frac{1-\beta+r-2w}{2(1-\beta)} = 0$, results in $w^* = \frac{1-\beta+r}{2}$, which, according to Lemma 1, must be greater than $\frac{2r}{\beta} - (1 - \beta + r)$, or $r < \frac{3\beta(1-\beta)}{4-3\beta} = \rho_1$, for this equilibrium to be valid.

For case (ii), $p^* = \frac{r}{\beta}$. The manufacturer, unwilling to leave money on the table, always chooses the highest value from the range $\frac{2r}{\beta} - 1 \leq w \leq \frac{2r}{\beta} - (1 - \beta + r)$, resulting in $w^* = \frac{2r}{\beta} - (1 - \beta + r)$. This equilibrium is valid across all $r \leq \beta$. Point to note Point to note here is that, if $r > \beta$, then r is also greater than ρ_2, which can be shown to be less than β. Therefore, $r > \beta$ falls under case (iii), the benchmark region, which we discuss next. Viewed differently, if $r > \beta$, then $p^* = \frac{r}{\beta} > 1$, and no consumer would buy the product. Therefore, $r > \beta$ cannot happen in case (ii).

Finally, in case (iii), $p^*(w) = \frac{1+w}{2}$, and the manufacturer's profit is

$$\pi_m = \frac{w(1-w)}{2} \tag{B4}$$

implying $w^* = \frac{1}{2}$. According to Lemma 1, this w^* must be less than $\frac{2r}{\beta} - 1$, or $r > \frac{3\beta}{4} = \rho_5$.

Since $\rho_1 < \rho_5$, (ii) is the only valid equilibrium if $\rho_1 \leq r \leq \rho_5$, and $w^* = \frac{2r}{\beta} - (1 - \beta + r)$. If $r < \rho_1$, the manufacturer can either set $w = \frac{1-\beta+r}{2}$, or set $w = \frac{2r}{\beta} - (1 - \beta + r)$. If the manufacturer chooses $w = \frac{1-\beta+r}{2}$, its profit is $\frac{(1-\beta+r)^2}{8(1-\beta)}$ from (B3). On the other hand, if it chooses $w = \frac{2r}{\beta} - (1 - \beta + r)$, its profit becomes $w\left(1 - \frac{r}{\beta}\right) = \frac{(\beta-r)(r-(1-\beta)(\beta-r))}{\beta^2}$. Between the two choices, the manufacturer chooses the one that yields a higher profit. It is easy to verify that, at $r = \rho_1$, both options yield the same profit, and for $r < \rho_1$, the first option is always better. Thus, if $r < \rho_1$, (i) is the equilibrium outcome and $w^* = \frac{1-\beta+r}{2}$.

If $r > \rho_5$, the manufacturer can either set $w = \frac{1}{2}$, or set $w = \frac{2r}{\beta} - (1 - \beta + r)$. If $w = \frac{1}{2}$, the manufacturer's profit is $\frac{1}{8}$ from (B4), and, if $w = \frac{2r}{\beta} - (1 - \beta + r)$, the profit becomes, as before, $\frac{(\beta-r)(r-(1-\beta)(\beta-r))}{\beta^2}$. Comparing these two profits, it is easy to verify that the manufacturer would choose the first option if $r \geq \frac{\beta(6-4\beta+\sqrt{2\beta})}{4(2-\beta)} = \rho_2$. Since $\rho_2 > \rho_5$ holds trivially, (iii) is the equilibrium outcome with $w^* = \frac{1}{2}$ if $r \geq \rho_2$. By the same logic, for $\rho_5 \leq r < \rho_2$, case (ii) is the equilibrium. It should now be clear from the preceding discussion that case (ii) is the equilibrium for the entire range $\rho_1 \leq r < \rho_2$.

With the closed-form solution for w^*, we can derive p^* from Lemma 1. ■

Proof of Proposition 2

Using p^* and w^* from Proposition 1, we can find the equilibrium profits for the manufacturer and retailer as $\pi_m^* = w^*\, q(p^*)$ and $\pi_r^* = (p^* - w^*)\, q(p^*)$, respectively. ■

Proof of Theorem 1

First, since in the piracy region $r < \rho_1$, the manufacturer's profit, $\pi_m^* = \frac{(1-\beta+r)^2}{8(1-\beta)}$, is increasing in r, equating this profit to the benchmark profit of $\pi_{m0} = \frac{1}{8}$ and solving for r, we find $r = \sqrt{1-\beta} - (1 - \beta)$; of course, for it to be a valid root this r must abide by the restriction $r < \rho_1$, which is equivalent to $\beta < \frac{8}{9}$. Next, in the threat region ($\rho_1 \leq r \leq \rho_2$), the manufacturer's profit, $\pi_m^* = \frac{(\beta-r)(r-(1-\beta)(\beta-r))}{\beta^2}$, can never be less than π_{m0}. In other words, for all $\beta < \frac{8}{9}$, a necessary and sufficient for the manufacturer to be better off is $\sqrt{1-\beta} - (1-\beta) < r < \rho_2$.

The case of $\beta > \frac{8}{9}$ is somewhat different. Here, the threat region takes over at a lower r; the profit function for the threat region meets the benchmark profit, $\pi_{m0} = \frac{1}{8}$, two times, first at point ρ_2^c and then again at ρ_2, where ρ_2^c is the root conjugate to ρ_2 and is given by

$$\rho_2^c = \frac{\beta(6-4\beta-\sqrt{2\beta})}{4(2-\beta)}$$

Therefore, for all $\beta > \frac{8}{9}$, the manufacturer would be better off if and only if $\rho_2^c < r < \rho_2$. Define

$$\rho_3 = \begin{cases} \sqrt{1-\beta} - (1-\beta), & \text{if } \beta \leq \frac{8}{9} \\ \rho_2^c = \frac{\beta(6-4\beta-\sqrt{2\beta})}{4(2-\beta)}, & \text{otherwise} \end{cases}$$

It is then immediate that the manufacturer is better off if $\rho_3 < r < \rho_2$.

Next, we consider the retailer. The retailer's profit, $\pi_r^* = \frac{(1-\beta+r)^2}{16(1-\beta)}$, is also increasing in r in the piracy region ($r < \rho_1$). Therefore, as before, equating this profit to the benchmark profit of $\pi_{r0} = \frac{1}{16}$ and solving for r, we find that the retailer would also be better off if $r > \sqrt{1-\beta} -$

$(1-\beta)$ and $\beta \leq \frac{8}{9}$. In the threat region ($\rho_1 \leq r < \rho_2$), the retailer's profit, $\pi_r^* = \frac{(1-\beta)(\beta-r)^2}{\beta^2}$, is decreasing in r. This profit is greater than or equal to π_{r0} if and only if $r < \beta\left(1-\frac{1}{4\sqrt{1-\beta}}\right)$ and $\beta \leq \frac{8}{9}$. We define

$$\rho_4 = \begin{cases} \beta\left(1-\frac{1}{4\sqrt{1-\beta}}\right), & \text{if } \beta \leq \frac{8}{9} \\ \rho_2^c = \frac{\beta(6-4\beta-\sqrt{2\beta})}{4(2-\beta)}, & \text{otherwise} \end{cases}$$

Clearly then, the retailer is better off in the presence of piracy or its threat if $\rho_3 < r < \rho_4$. The three regions in the theorem then emerge by combining the above. ■

Proof of Proposition 3

The consumer surplus (CS) for all consumers, legal and illegal, can be found by aggregating their consumption benefits net of the price they pay or the penalty they incur. Therefore, CS is given by

$$CS = \begin{cases} \displaystyle\int_{\frac{p^*-r}{1-\beta}}^{1} (v-p^*)dv + \underbrace{\int_{\frac{r}{\beta}}^{\frac{p^*-r}{1-\beta}} (\beta v - r)dv}_{\text{Pirate Surplus}}, & \text{if } p^* \geq \frac{r}{\beta} \\ \displaystyle\int_{p^*}^{1} (v-p^*)dv, & \text{otherwise} \end{cases}$$

The desired result can now be obtained by algebraic manipulation after substituting p^* from (2) into the expression above.

The above expression includes the net surplus from the legal users as well as that from the pirates. If one is interested in finding the consumer surplus excluding that of the pirates, it can be easily accomplished by dropping the term marked as "Pirate Surplus" above. ■

Proof of Theorem 2

In the piracy region, $CS = \frac{1+15\beta-30r}{32} + \frac{r^2}{32}\left(\frac{1}{1-\beta}+\frac{16}{\beta}\right)$. Its derivative, $\frac{\partial(CS)}{\partial r} = r\left(\frac{1}{16(1-\beta)}+\frac{1}{\beta}\right) - \frac{15}{16}$ is an increasing function of r. However, since $r < \rho_1$ in the piracy region, we must have

$$\frac{\partial(CS)}{\partial r} < \rho_1\left(\frac{1}{16(1-\beta)}+\frac{1}{\beta}\right) - \frac{15}{16} = -\frac{3}{4(4-3\beta)} < 0$$

In other words, in the piracy region, CS is decreasing in r, and is minimized at $r = \rho_1$. Now, $CS|_{r=\rho_1} = \frac{1}{2(4-3\beta)^2} > \frac{1}{32}$. Clearly then, CS in the piracy region is always above the benchmark value of $CS_0 = \frac{1}{32}$.

Furthermore, the consumer surplus in the threat region, $CS = \frac{(\beta-r)^2}{2\beta^2}$ is decreasing in r. Therefore, by equating it to CS_0, we find that consumers are better off if $r < \rho_5 = \frac{3\beta}{4}$.

Since $\rho_4 < \rho_5$ for all $\beta > 0$, the result follows from Theorem 1. ■

Proof of Proposition 4

Since the channel profit is given by $CP = \pi_m^* + \pi_r^*$, it can be easily calculated from Proposition 2. Further, social welfare can be calculated from

$$SW = \begin{cases} \displaystyle\int_{\frac{p^*-r}{1-\beta}}^{1} v\,dv + \underbrace{\int_{\frac{r}{\beta}}^{\frac{p^*-r}{1-\beta}} \beta v\,dv}_{\text{Welfare from Piracy}}, & \text{if } p^* \geq \frac{r}{\beta} \\ \displaystyle\int_{p^*}^{1} v\,dv, & \text{otherwise} \end{cases}$$

Substituting p^* from (2) into the above expression, we get the desired result. Of course, if one is interested in calculating the social surplus without including the pirates, it can be easily done by dropping the term labeled "Welfare from Piracy" above. ■

Proof of Theorem 3

In the piracy region, $CP = \frac{3(1-\beta+r)^2}{16(1-\beta)}$ is increasing in r. Equating it to CP_0, we get $r = \sqrt{1-\beta} - (1-\beta)$, which is valid only if it is less than ρ_1, or equivalently, if $\beta < \frac{8}{9}$.

Now, in the threat region, $CP = \frac{r(\beta-r)}{\beta^2}$ is concave in r. Equating it to CP_0, we get two roots, $r = \rho_5^c = \frac{\beta}{4}$ and $r = \rho_5 = \frac{3\beta}{4}$. The first root is less than ρ_1 and the second greater; as long as r is between these two roots, CP is higher than its benchmark. Defining

$$\rho_6 = \begin{cases} \sqrt{1-\beta} - (1-\beta), & \text{if } \beta \leq \frac{8}{9} \\ \dfrac{\beta}{4}, & \text{otherwise} \end{cases}$$

we conclude that channel profit is higher if $\rho_6 < r < \rho_5$.

As far as social welfare is concerned, in the piracy region, $SW = \frac{7+9\beta+6r}{32} - \frac{r^2}{32}\left(\frac{1}{1-\beta} + \frac{16}{\beta}\right)$ is clearly concave in r. Therefore, the minimum value of SW occurs at one of the extremes, that is either at $r = 0$ or at $r = \rho_1$. Both these extreme values of SW can be easily shown to be greater than SW_0, implying that piracy always leads to a higher social surplus. We now move to the threat region, where $SW = \frac{1}{2}\left(1 - \frac{r^2}{\beta^2}\right)$ is clearly decreasing in r. Equating it to SW_0, we find that the threat region does better in terms of social welfare, if $r < \rho_5 = \frac{3\beta}{4}$. This completes the proof. ■

Proof of Proposition 5

When $\bar{p} > \frac{r}{\beta}$, $\bar{\pi} = \bar{p}q(\bar{p}) = \bar{p}\left(1 - \frac{\bar{p}-r}{1-\beta}\right)$, implying $\frac{\partial\bar{\pi}}{\partial\bar{p}} = 1 - \frac{2\bar{p}-r}{1-\beta}$.

Since $\frac{\partial^2\bar{\pi}}{\partial\bar{p}^2} = -\frac{2}{1-\beta} < 0$, the first order condition, $\frac{\partial\bar{\pi}}{\partial\bar{p}} = 0$, resuls in $\bar{p}^* = \frac{1}{2}(1-\beta+r)$. Clearly, this solution must be greater than $\frac{r}{\beta}$, or $r < \frac{(1-\beta)\beta}{2-\beta} = \bar{\rho}_1$.

If, on the other hand, $\bar{p} < \frac{r}{\beta}$, then $\bar{\pi} = \bar{p}q(\bar{p}) = \bar{p}(1 - \bar{p})$, resulting in $\bar{p}^* = \frac{1}{2}$. This $\bar{p}^*$ should be less than or equal to $\frac{r}{\beta}$, implying $r \geq \frac{\beta}{2} = \bar{\rho}_2$.

Now, $\bar{\rho}_2 - \bar{\rho}_1 = \frac{\beta^2}{2(2-\beta)}$, that is, $\bar{\rho}_1 < \bar{\rho}_2$. Therefore, we also consider the situation where $\bar{\rho}_1 \le r < \bar{\rho}_2$. In that situation, the profit above is decreasing for $\bar{p} > \frac{r}{\beta}$ but increasing for $\bar{p} < \frac{r}{\beta}$. So, $\bar{p}^*$ becomes $\frac{r}{\beta}$.

The optimal profit in each region can be found easily from $\bar{p}^* q(\bar{p}^*)$. ■

Proof of Theorem 4

We start by noting that, in the benchmark region, where neither piracy nor its threat is present, $\eta = \frac{3/16}{1/4} = \frac{3}{4}$. We will now show that, for $\bar{\rho}_1 < r < \rho_5$, η is larger than $\frac{3}{4}$. To do so, we make use of Proposition 4 and Proposition 5. This allows us to divide the interval $(\bar{\rho}_1, \rho_5)$ into several parts:

- When $\bar{\rho}_1 < r < \rho_1$, $CP = \frac{3(1-\beta+r)^2}{16(1-\beta)}$ and $\bar{\pi}^* = \frac{r(\beta-r)}{\beta^2}$. Therefore, $\eta = \frac{3\beta^2 (1-\beta+r)^2}{16r(1-\beta)(\beta-r)}$, and $\eta - \frac{3}{4} = \frac{3\left(\beta(1-\beta)-r(2-\beta)\right)^2}{16r(1-\beta)(\beta-r)} > 0$.
- If $\rho_1 \le r \le \bar{\rho}_2$, $CP = \bar{\pi}^* = \frac{r(\beta-r)}{\beta^2}$. Therefore, $\eta = 1$, and the channel is fully coordinated.
- Finally, when $\bar{\rho}_2 < r < \rho_5$, $CP = \frac{r(\beta-r)}{\beta^2}$ and $\bar{\pi}^* = \frac{1}{4}$. Therefore, $\eta = \frac{4r(\beta-r)}{\beta^2}$, which is greater than $\frac{3}{4}$ because $r < \rho_5 = \frac{3\beta}{4}$. ■

Proof of Lemma 2

It is easy to verify that the manufacturer's profit is increasing in r in the piracy region, but concave in the threat region, implying that the maximum must happen in the threat region. The manufacturer's profit in the threat region is given by

$$\pi_m^* = \frac{(\beta - r)(r - (1-\beta)(\beta - r))}{\beta^2}$$

Since $\frac{\partial^2 \pi_m^*}{\partial r^2} = -\frac{2(2-\beta)}{\beta^2} < 0$, we simply solve $\frac{\partial \pi_m^*}{\partial r} = 0$ to obtain $r_m^* = \frac{\beta(3-2\beta)}{4-3\beta}$.

As far as the retailer is concerned, it can be easily verified that its profit is increasing in r in the piracy region and decreasing in the threat region. Therefore, it is maximized at ρ_1, implying $r_r^* = \rho_1 = \frac{3\beta(1-\beta)}{4-3\beta}$. Of course, the profit at r_r^* can be better than the benchmark profit only if $\beta < \frac{8}{9}$. If $\beta \ge \frac{8}{9}$, however, the retailer would prefer an r that is greater than ρ_2.

It is also easy to verify that the channel profit is increasing in r in the piracy region, but concave in the threat region. The channel profit, *CP*, in the threat region is $\frac{r(\beta-r)}{\beta^2}$, which is maximized at $r_C^* = \frac{\beta}{2}$.

Consumer surplus is always decreasing in r, implying that the maximum occurs at r_C^*.

Finally, the total social welfare, SW, is concave in r in the piracy region, but decreasing in the threat region. Now, in the piracy region

$$SW = \frac{7 + 9\beta + 6r}{32} - \frac{r^2}{32}\left(\frac{1}{1-\beta} + \frac{16}{\beta}\right)$$

Since $\frac{\partial^2 (SW)}{\partial r^2} = -\left(\frac{1}{16(1-\beta)} + \frac{1}{\beta}\right) < 0$, we can solve $\frac{\partial (SW)}{\partial r} = 0$ to get $r_S^* = \frac{3\beta(1-\beta)}{16-15\beta}$. ■

Proof of Proposition 6

To prove this result, we need to show that $r_C^*, r_S^* < r_r^*, r_c^*, r_m^*$. Now it can be easily shown that $r_r^* < r_c^* < r_m^*$. Further, because $r_C^* = 0$, $r_C^* < r_S^*$ holds trivially. Therefore the proof can be completed by simply showing that $r_S^* < r_r^*$. Now,

$$r_S^* - r_r^* = \frac{\beta(36 - 59\beta + 24\beta^2)}{64 - 92\beta + 30\beta^2} > 0, \forall \beta \in (0,1)$$

which completes the proof. ■