Designing Real-Time Feedback for Bidders In Homogeneous-Item Continuous Combinatorial Auctions

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Appendix A

Proofs of all Theorems, Corollaries, and Lemmas for SIMU-OR Auctions

Theorem 1. For a SIMU-OR auction, given auction state k and new bid b_{k+1}:

1. $(\forall x < s(b_{k+1}))(WIN_{k+1}[x] = WIN_k[x])$;
2. $(\forall x \geq s(b_{k+1}))(WIN_{k+1}[x] = max_{\prec}[WIN_k[x], \{b_{k+1}\} \cup WIN_k[x - s(b_{k+1})]])$

Proof. The first statement follows immediately from the definition of $WIN_{k+1}[x]$, because $b_{k+1} \notin WIN_{k+1}[x] \Rightarrow WIN_{k+1}[x] = WIN_k[x]$. The second statement immediately follows from two facts: (a) the only way to have $WIN_{k+1}[x] \neq WIN_k[x]$ is when $b_{k+1} \in WIN_{k+1}[x]$; and (b) since bid b_{k+1} "covers" only $s(b_{k+1})$ items, the best possible allocation among $C_{k+1}[x]$ involving b_{k+1} should also involve the best possible prior bids (i.e., from B_k) covering the remaining $x - s(b_{k+1})$ items. Hence, the only alternative to $WIN_k[x]$ would be $\{b_{k+1}\} \cup WIN_k[x - s(b_{k+1})]$. ■

Theorem 2. For a SIMU-OR auction, given auction state k and new bid b_{k+1},

$$b_{k+1} \in WIN_{k+1} \Leftrightarrow v(b_{k+1}) > REV_k(N) - REV_k(N - s(b_{k+1}))$$

Proof. Immediate from Theorem 1, since $b_{k+1} \in WIN_{k+1}$ if and only if $WIN_k \prec \{b_{k+1}\} \cup WIN_k[N - s(b_{k+1})]$. Based on the definition of strict total order $\prec$ and taking into account that WIN_k chronologically precedes allocation $\{b_{k+1}\} \cup WIN_k[N - s(b_{k+1})]$, we have that $v(b_{k+1}) > REV_k(N) - REV_k(N - s(b_{k+1}))$. ■

Corollary 2a. For a SIMU-OR auction, given auction state k, the winning level at span x is calculated as $WL_k(x) = REV_k(N) - REV_k(N - x)$

Proof. Immediate from Theorem 2 and the definition of winning level. ■

Corollary 2b. For a SIMU-OR auction, given auction state k and new bid b_{k+1}:

$$b_{k+1} \in WIN_{k+1}[i] \Leftrightarrow v(b_{k+1}) > REV_k(i) - REV_k(i - s(b_{k+1}))$$

Proof. Immediate from Theorem 2, by considering sub-auction i. ■

Theorem 3. For a SIMU-OR auction, given auction state *k* and any bid $b \in B_k$:
$$b \in LIVE_k \Leftrightarrow \big(\exists x \geq s(b)\big)(b \in WIN_k[x])$$

Proof. First, we prove $\big(\exists x \geq s(b)\big)(b \in WIN_k[x]) \Rightarrow b \in LIVE_k$. Consider two following cases for *x*:
[**Case 1**: $x = N$] $b \in WIN_k[N] \Rightarrow b \in WIN_k \Rightarrow b \in LIVE_k$.
[**Case 2**: $x < N$] Consider new bid b_{k+1}, such that $s(b_{k+1}) = N - x$ and $v(b_{k+1}) > REV_k - v(WIN_k[x])$. Furthermore, let's consider allocation $\alpha = \{b_{k+1}\} \cup WIN_k[x]$. By definition, $s(\alpha) = s(b_{k+1}) + s(WIN_k[x]) \leq N - x + x = N$ and $v(\alpha) = v(b_{k+1}) + v(WIN_k[x]) > REV_k$. I.e., $s(\alpha) \leq N$ and $v(\alpha) > REV_k$. Hence, $\alpha = WIN_{k+1}$. Because $b \in WIN_k[x]$ and $WIN_k[x] \subseteq \alpha$, we have that $b \in WIN_{k+1}$. Consequently, $b \in LIVE_k$.

Next, we prove $b \in LIVE_k \Rightarrow \big(\exists x \geq s(b)\big)(b \in WIN_k[x])$. From $b \in LIVE_k$ we have that there exists an auction state *l* ($l \geq k$) such that $b \in WIN_l$. Denote WIN_l as: $WIN_l = \{b\} \cup \beta_{1,k} \cup \beta_{k+1,l}$, where $\beta_{1,k} = \{b' \in B_k \wedge b' \neq b\}$ and $\beta_{k+1,l} = \{b' \in WIN_l | b' \notin B_k\}$. Thus, bid sets $\{b\}$, $\beta_{1,k}$, and $\beta_{k+1,l}$ are pairwise disjoint. Suppose otherwise, i.e., $\big(\forall x \geq s(b)\big)(b \notin WIN_k[x])$. Consider bid set $\alpha = \{b\} \cup \beta_{1,k}$. Denote $x' = s(\alpha) = s(b) + s(\beta_{1,k})$. Since $x' \geq s(b)$, we have that $b \notin WIN_k[x']$. Furthermore, by definition, $\alpha \subseteq B_k[x']$ and $b \in \alpha$. Therefore, $\alpha \prec WIN_k[x']$. Consequently, $WIN_l = \alpha \cup \beta_{k+1,l} \prec WIN_k[x'] \cup \beta_{k+1,l}$, i.e., we have found a set of bids $WIN_k[x'] \cup \beta_{k+1,l} \subseteq B_l$ that is better than WIN_l, which is a contradiction to the definition of WIN_l. ■

Corollary 3a. For a SIMU-OR auction, given auction state *k*, the deadness level at span *x* is calculated as $DL_k(x) = \boldsymbol{min}_{i \in \{x,\ldots,N\}}[REV_k(i) - REV_k(i-x)]$

Proof. Immediate from Theorem 3 and Corollary 2b. I.e., let $s(b_{k+1}) = x$, then $b_{k+1} \in LIVE_{k+1} \Leftrightarrow (\exists i \geq x)(b_{k+1} \in WIN_{k+1}[i]) \Leftrightarrow (\exists i \geq x)\big(v(b_{k+1}) > REV_k(i) - REV_k(i-x)\big) \Leftrightarrow v(b_{k+1}) > \boldsymbol{min}_{i \in \{x,\ldots,N\}}[REV_k(i) - REV_k(i-x)]$. The results follow based on the definition of deadness level.■

Theorem 4. For a SIMU-OR auction, given any auction state *k* and sub-auctions *x* and *y*, the following statements are true:
1. $DL_k(x) \leq REV_k(x)$ 2. $DL_k(x) \leq WL_k(x)$ 3. $DL_k(N) = WL_k(N) = REV_k(N)$
4. $DL_k(x) \leq DL_{k+1}(x)$ 5. $x \leq y \Rightarrow DL_k(x) \leq DL_k(y)$ and $WL_k(x) \leq WL_k(y)$
6. $b \in WIN_k \Rightarrow WL_k\big(s(b)\big) \leq v(b)$ 7. $b \in LIVE_k \Rightarrow DL_k\big(s(b)\big) \leq v(b)$

Proof. Statements 1-7 follow immediately from the definitions of $DL_k(x)$ and $WL_k(x)$ as well as the definition and properties of $REV_k(x)$. ■

Theorem 5. In a SIMU-OR auction of size *N*, for any auction state *k* we have: $|LIVE_k| \leq N$.

Proof. Based on Corollary 3b, we have that $LIVE_k = \bigcup_{1 \leq i \leq N} WIN_k[i]$. Therefore, instead of analyzing set $LIVE_k$**,** we will focus on the set $\bigcup_{1 \leq i \leq N} WIN_k[i]$. Specifically, we will show that $|\bigcup_{1 \leq i \leq x} WIN_k[i]| \leq x$ for all $x = 1, \ldots, N$. In other words, given any $x = 1, \ldots, N$, we will show that, if we consider *all* sub-auctions from 1 to *x*, no more than *x* different bids can appear in the winning allocations of these sub-auctions.

We will prove that $|\bigcup_{1 \leq i \leq x} WIN_k[i]| \leq x$ by induction on *x*. Obviously, the base case *x* = 1 holds, since $WIN_k[1]$ is either a singleton set (i.e., it represents the largest-valued 1-item bid that was submitted to the auction) or an empty set (if no 1-item bids were submitted so far). In other words, $|WIN_k[1]| \leq 1$. Now let's assume that this statement holds for *x*, i.e., $|\bigcup_{1 \leq i \leq x} WIN_k[i]| \leq x$, and prove that it then must hold for $x+1$ as well, i.e., $|\bigcup_{1 \leq i \leq x+1} WIN_k[i]| \leq x+1$.

Let's assume otherwise, i.e., $|\bigcup_{1 \leq i \leq x+1} WIN_k[i]| \geq x+2$, which means that there must exist at least two bids $b' \in WIN_k[x+1]$ and $b'' \in WIN_k[x+1]$, such that $b' \notin \bigcup_{1 \leq i \leq x} WIN_k[i]$ and $b'' \notin \bigcup_{1 \leq i \leq x} WIN_k[i]$. Let's denote the spans of b' and b'' as s' and s'' (obviously, $s', s'' \geq 1$), respectively, and rewrite $WIN_k[x+1]$ as $WIN_k[x+1] = \{b'\} \cup \{b''\} \cup WIN_k[x+1-$

$s' - s'']$, where $WIN_k[x + 1 - s' - s'']$ denotes the "rest" of $WIN_k[x + 1]$, besides b' and b''. We can write this way for 3 reasons: (a) we know that b' and b'' belong to $WIN_k[x + 1]$; (b) $WIN_k[x + 1]$ should involve the *best* possible allocation that can cover the remaining $x + 1 - s' - s''$ items of sub-auction $x + 1$, not covered by b' and b'', hence $WIN_k[x + 1 - s' - s'']$; (c) since $b', b'' \notin \bigcup_{1 \le i \le x} WIN_k[i]$, we have that $b', b'' \notin WIN_k[x + 1 - s' - s'']$, i.e., we are not "double counting" b' and b''.

Now consider the following allocation $C' = \{b'\} \cup WIN_k[x + 1 - s' - s'']$. Obviously, $s(C') \le x + 1 - s''$. Also, since $b' \in C'$ and $b' \notin WIN_k[x + 1 - s'']$, we have that $C' \ne WIN_k[x + 1 - s'']$. Or, more precisely, $C' \prec WIN_k[x + 1 - s'']$. Finally, we go back to the expression for $WIN_k[x + 1]$ and plug in the results, i.e., $WIN_k[x + 1] = \{b'\} \cup \{b''\} \cup WIN_k[x + 1 - s' - s''] = C' \cup \{b''\} \prec WIN_k[x + 1 - s''] \cup \{b''\}$. Note that there is no danger of "double counting" in the latest expression either, since $b'' \notin WIN_k[x + 1 - s'']$. Therefore, we derive that $WIN_k[x + 1] \prec WIN_k[x + 1 - s''] \cup \{b''\}$, i.e., there exists an allocation with the span less than or equal $x + 1$ that is better than the best allocation with that span, $WIN_k[x + 1]$. Contradiction. Therefore, our assumption that $|\bigcup_{1 \le i \le x+1} WIN_k[i]| \ge x + 2$, was incorrect. This leads us to the result that $|\bigcup_{1 \le i \le x+1} WIN_k[i]| \le x + 1$, which completes the proof by induction.

Therefore, we have $|\bigcup_{1 \le i \le x} WIN_k[i]| \le x$ for all $x = 1, \dots, N$. The proof of the theorem is concluded by choosing $x = N$, i.e., $|LIVE_k| = |\bigcup_{1 \le i \le N} WIN_k[i]| \le N$. ■

Appendix B

Proofs of all Theorems, Corollaries, and Lemmas for SIMU-XOR Auctions

Theorem 6. For a SIMU-XOR auction, given auction state k and new bid $b_{k+1} = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$, $\forall x \in \{1,2, \dots, N\}$, $\forall P \subseteq \mathbb{P}$:

1. If $p(b_{k+1}) \notin P$, then $WIN_{k+1}[x, P] = WIN_k[x, P]$.
2. If $p(b_{k+1}) \in P$, then

$WIN_{k+1}[x, P] = max_{\prec}\{WIN_k[x, P], WIN_k[x - 1, P \backslash \{p(b_{k+1})\}] \cup \{\sigma_1\}, WIN_k[x - 2, P \backslash \{p(b_{k+1})\}] \cup \{\sigma_2\}, \dots, WIN_k[1, P \backslash \{p(b_{k+1})\}] \cup \{\sigma_{x-1}\}, \{\sigma_x\}\}$.

Proof. Let $p^* = p(b_{k+1})$. The first statement follows immediately from the definition of $WIN_{k+1}[x, P]$. I.e., $p^* \notin P \Rightarrow \forall \sigma \in b_{k+1}, \sigma \notin WIN_{k+1}[x, P] \Rightarrow WIN_{k+1}[x, P] = WIN_k[x, P]$. The second statement immediately follows from two facts: (a) the only way to have $WIN_{k+1}[x, P] \ne WIN_k[x, P]$ is when $\exists \sigma \in b_{k+1}, \sigma \in WIN_{k+1}[x, P]$; and (b) since any atomic bid σ "covers" only $s(\sigma)$ items, the best allocation among $C_{k+1}[x, P]$ involving σ should also involve the best possible prior bids (i.e., from B_k) covering the remaining $x - s(\sigma)$ items, made by the set of bidders excluding $p(\sigma) = p^*$. Hence, the only alternative to $WIN_k[x, P]$ is $WIN_k[x - s(\sigma), P \backslash \{p^*\}] \cup \{\sigma\}$. Following the same logic, when we consider a new *general* bid $b = (\sigma_1, \sigma_2, \dots, \sigma_N)$, the possible alternatives to $WIN_k[x, P]$ include $WIN_k[x - 1, P \backslash \{p^*\}] \cup \{\sigma_1\}$, $WIN_k[x - 2, P \backslash \{p^*\}] \cup \{\sigma_2\}$, …, $WIN_k[1, P \backslash \{p^*\}] \cup \{\sigma_{x-1}\}$, and $\{\sigma_x\}$, and the exact winning allocation is determined by total order $\prec$ that satisfies allocative fairness. ■

Theorem 7. For a SIMU-XOR auction, given auction state k and new atomic bid σ. Then $\sigma \in WIN_{k+1} \Leftrightarrow v(\sigma) > REV_k(N, \mathbb{P}) - REV_k(N - s(\sigma), \mathbb{P} \backslash \{p(\sigma)\})$

Proof. Let $s(\sigma) = x$ and $p(\sigma) = p^*$. Immediate from Theorem 6, because $\sigma \in WIN_{k+1}$ if and only if $WIN_k \prec \{\sigma\} \cup WIN_k[N - x, \mathbb{P} \backslash \{p^*\}]$. Based on the definition of strict total order $\prec$ and the fact that WIN_k chronologically precedes allocation $\{\sigma\} \cup WIN_k[N - x, \mathbb{P} \backslash \{p^*\}]$, we have $v(\sigma) > REV_k(N, \mathbb{P}) - REV_k(N - x, \mathbb{P} \backslash \{p^*\})$. ■

Corollary 7a. For a SIMU-XOR auction, given auction state k, the winning level at span x for bidder p^* is calculated as $WL_k(x, p^*) = REV_k(N, \mathbb{P}) - REV_k(N - x, \mathbb{P} \backslash \{p^*\})$

Proof. Immediate from Theorem 7 and the definition of winning level. ■

Corollary 7b. For a SIMU-XOR auction, given auction state k and new atomic bid σ. For all bidder set P such that $p(\sigma) \in P \subseteq \mathbb{P}$

$$\sigma \in WIN_{k+1}[i, P] \Leftrightarrow v(\sigma) > REV_k(i, P) - REV_k(i - s(\sigma), P \backslash \{p(\sigma)\})$$

Proof. Immediate from Theorem 7, by considering sub-auction $[i, P]$. ■

Theorem 8. For a SIMU-XOR auction, given auction state k and any atomic bid $\sigma \in B_k$.

1. If $s(\sigma) \leq N - |\mathbb{P}|$, then: $\sigma \in LIVE_k \Leftrightarrow \sigma = WIN_k[s(\sigma), p(\sigma)]$;
2. If $s(\sigma) > N - |\mathbb{P}|$, then: $\sigma \in LIVE_k \Leftrightarrow \exists Q \subseteq \mathbb{P}$ with $p(\sigma) \in Q$ and $|Q| = |\mathbb{P}| - \big(N - s(\sigma)\big)$ such that $\sigma = WIN_k[s(\sigma), Q]$

Proof. [Case 1] Note that, if $s(\sigma) \leq N - |\mathbb{P}|$, then the atomic bid is automatically live, as long as it is not outbid by other bids submitted by the same bidder with same or smaller span (i.e., $\sigma = WIN_k[s(\sigma), p(\sigma)]$). This is because, for an arbitrary atomic bid σ, we can easily construct a "complementary" future allocation β which satisfies $s(\beta) = N - s(\sigma)$, $p(\beta) = \mathbb{P} \backslash \{p(\sigma)\}$, and $v(\beta) > REV_k(N, \mathbb{P}) - REV_k\big(s(\sigma), p(\sigma)\big)$. It follows that $\{\sigma\} \cup \beta = WIN_{t(\beta)}$ and $\sigma \in LIVE_k$.

[Case 2] We first prove $\exists Q \subseteq \mathbb{P}$ with $p(\sigma) \in Q$ and $|Q| = |\mathbb{P}| - \big(N - s(\sigma)\big)$ such that $\sigma = WIN_k[s(\sigma), Q] \Rightarrow \sigma \in LIVE_k$. If $s(\sigma) = N$, then $N - s(\sigma) = 0$ and $|Q| = |\mathbb{P}|$, which implies that $WIN_k[s(\sigma), Q] = WIN_k$. Therefore, $\sigma = WIN_k[s(\sigma), Q] \Rightarrow \sigma = WIN_k \Rightarrow \sigma \in LIVE_k$. If instead $s(\sigma) < N$, consider a future allocation β where $s(\beta) = N - s(\sigma)$, $|p(\beta)| = N - s(\sigma)$, and $p(\beta) \cap Q = \emptyset$. In other words, β covers $N - s(\sigma)$ items and contains bidders that are not in Q. Suppose $v(\beta) > REV_k(N, \mathbb{P}) - REV_k(x, Q)$, then $WIN_k[s(\sigma), Q] \cup \beta = WIN_{t(\beta)}$. Therefore, $\sigma \in LIVE_k$.

We then prove $\sigma \in LIVE_k \Rightarrow \exists Q \subseteq \mathbb{P}$ with $p(\sigma) \in Q$ and $|Q| = |\mathbb{P}| - \big(N - s(\sigma)\big)$ such that $\sigma = WIN_k[s(\sigma), Q]$. Instead of directly proving this, we prove the contrapositive statement, i.e., $\forall Q \subseteq \mathbb{P}$ with $p(\sigma) \in Q$ and $|Q| = |\mathbb{P}| - \big(N - s(\sigma)\big)$, $\sigma \neq WIN_k[s(\sigma), Q] \Rightarrow \sigma \in DEAD_k$. Consider an allocation α of the whole auction that contains σ, i.e., $s(\alpha) \leq N$ and $\sigma \in \alpha$. Let $\beta = \alpha \backslash \{\sigma\}$, thus, we know that $s(\beta) \leq N - s(\sigma)$. Because each bidder in an allocation has to bid on at least 1 item, we have $|p(\beta)| \leq N - s(\sigma)$, or $|\mathbb{P}| - |p(\beta)| \geq |\mathbb{P}| - \big(N - s(\sigma)\big)$. Consider $WIN_k[s(\sigma), \mathbb{P} \backslash p(\beta)]$. Because $\beta = \alpha \backslash \{\sigma\}$, we know $p(\sigma) \in \mathbb{P} \backslash p(\beta)$. Therefore, there exists $Q \subseteq \mathbb{P} \backslash p(\beta)$ that satisfies $|Q| = |\mathbb{P}| - (N - s(\sigma))$ and $p(\sigma) \in Q$. We either have $WIN_k[s(\sigma), Q] \prec WIN_k[s(\sigma), \mathbb{P} \backslash p(\beta)]$, or $WIN_k[s(\sigma), Q] = WIN_k[s(\sigma), \mathbb{P} \backslash p(\beta)]$. Furthermore, for any bidder set Q such that $|Q| = |\mathbb{P}| - (N - x)$ and $p(\sigma) \in Q$, $\{\sigma\}$ itself is a feasible allocation for sub-auction $[s(\sigma), Q]$. Thus, the fact that $\sigma \neq WIN_k[s(\sigma), Q]$ implies $\{\sigma\} \prec WIN_k[s(\sigma), Q]$. Therefore, we always have $\{\sigma\} \prec WIN_k[s(\sigma), \mathbb{P} \backslash p(\beta)]$. As a result, $\alpha = \beta \cup \{\sigma\} \prec \beta \cup WIN_k[s(\sigma), \mathbb{P} \backslash p(\beta)]$, and therefore $\sigma \in DEAD_k$. ■

Corollary 8a. For a SIMU-XOR auction, given auction state k, the deadness level at span x for bidder p^* is calculated as follows:

1. If $x \leq N - |\mathbb{P}|$, then: $DL_k(x, p^*) = REV_k(x, p^*)$;
2. If $x > N - |\mathbb{P}|$, then: $DL_k(x, p^*) = \min\limits_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N - x)} REV_k(x, Q)$

Proof. Consider a new atomic bid, σ, to be submitted at state $k+1$. Let $s(\sigma) = x$ and $p(\sigma) = p^*$. If $x \leq N - |\mathbb{P}|$, then based on Theorem 8, $\sigma \in LIVE_{k+1} \Leftrightarrow \sigma = WIN_{k+1}[x, p^*] \Leftrightarrow WIN_k[x, p^*] \prec \{\sigma\} \Leftrightarrow v(\sigma) > REV_k(x, p^*) \Leftrightarrow DL_k(x, p^*) = REV_k(x, p^*)$.

If instead $x > N - |\mathbb{P}|$, then based on Theorem 7, $\sigma \in LIVE_{k+1} \Rightarrow \exists P \subseteq \mathbb{P}$ with $p^* \in P$ and $|P| = |\mathbb{P}| - (N - x)$ such that $\sigma = WIN_{k+1}[x, P]$. Because $WIN_k[x, P]$ precedes $WIN_{k+1}[x, P]$ in time, it follows that $v(\sigma) = REV_{k+1}(x, P) > REV_k(x, P) \geq \min\limits_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N - x)} REV_k(x, Q)$. Therefore, we know that if σ is live, then $v(\sigma) > \min\limits_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N - x)} REV_k(x, Q)$. Conversely, if $v(\sigma) > \min\limits_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N - x)} REV_k(x, Q)$, suppose

$\min_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N-x)} REV_k(x, Q) = REV_k(x, P)$ where $p^* \in P$ and $|P| = |\mathbb{P}| - (N - x)$, then $WIN_k[x, P] \prec \{\sigma\}$. One can construct a "complementary" future bid combination β such that $s(\beta) = N - x$, $p(\beta) = \mathbb{P} \backslash P$. Then, $\beta \cup WIN_k[x, P] \prec \beta \cup \{\sigma\}$, and therefore σ is live. Together, it follows that $\sigma \in LIVE_{k+1} \Leftrightarrow v(\sigma) > \min_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N-x)} REV_k(x, Q) \Leftrightarrow DL_k(x, p^*) = \min_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N-x)} REV_k(x, Q)$. ■

Theorem 9. In a SIMU-XOR auction, for any auction state k, the following statements are true:
1. $DL_k(x, p) \leq REV_k(x, \mathbb{P})$
2. $DL_k(x, p) \leq WL_k(x, p)$
3. $\forall p \in \mathbb{P}$, $DL_k(N, p) = WL_k(N, p) = REV_k(N, \mathbb{P})$
4. $DL_k(x, p) \leq DL_{k+1}(x, p)$
5. $x \leq y \Rightarrow WL_k(x, p) \leq WL_k(y, p)$ and $DL_k(x, p) \leq DL_k(y, p)$
6. $\sigma \in WIN_k \Rightarrow WL_k(s(\sigma), p(\sigma)) = v(\sigma)$
7. $\sigma \in LIVE_k \Rightarrow DL_k(s(\sigma), p(\sigma)) = v(\sigma)$

Proof. Statements 1-5 follow immediately from the definitions of $DL_k(x, p)$, $WL_k(x, p)$, and $REV_k(x, p)$.
For statement 6, denote $WIN_k = \{\sigma\} \cup \beta$ where $s(\beta) \leq N - s(\sigma)$ and $p(\sigma) \notin p(\beta)$. By definition, $v(\sigma) + v(\beta) = REV_k(N, \mathbb{P}) = WL_k(s(\sigma), p(\sigma)) + REV_k(N - s(\sigma), \mathbb{P} \backslash \{p(\sigma)\})$. If $v(\beta) < REV_k(N - s(\sigma), \mathbb{P} \backslash \{p(\sigma)\})$, there must be another allocation β' with $s(\beta') \leq N - s(\sigma)$ and $p(\sigma) \notin p(\beta')$ that has $v(\beta') > v(\beta)$. As a result, $WIN_k \prec \{\sigma\} \cup \beta'$, contradiction. Therefore, $v(\beta) = REV_k(N - s(\sigma), \mathbb{P} \backslash \{p(\sigma)\})$ and $v(\sigma) = WL_k(s(\sigma), p(\sigma))$.

For statement 7, based on Theorem 8, if $s(\sigma) \leq N - |\mathbb{P}|$, then $\sigma \in LIVE_k \Rightarrow \sigma = WIN_k[s(\sigma), p(\sigma)] \Rightarrow v(\sigma) = REV_k(s(\sigma), p(\sigma))$. Meanwhile, Corollary 8a states that in this case, $DL_k(s(\sigma), p(\sigma)) = REV_k(s(\sigma), p(\sigma))$. Therefore, $DL_k(s(\sigma), p(\sigma)) = v(\sigma)$. Instead, if $s(\sigma) > N - |\mathbb{P}|$, based on Theorem 8, $\sigma \in LIVE_k \Rightarrow \sigma = WIN_k[s(\sigma), P]$ where $p(\sigma) \in P \subseteq \mathbb{P}$ and $|P| = |\mathbb{P}| - (N - s(\sigma))$, which implies $v(\sigma) = REV_k(s(\sigma), P)$. Meanwhile, Corollary 8a states that in this case, $DL_k(s(\sigma), p(\sigma)) = \min_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N - s(\sigma))} REV_k(s(\sigma), Q)$. Since $p(\sigma) \in Q$, we know that $REV_k(s(\sigma), Q) \geq v(\sigma)$. Overall, $v(\sigma) = REV_k(s(\sigma), P) = \min_{\forall Q \subseteq \mathbb{P}, p^* \in Q, |Q| = |\mathbb{P}| - (N - s(\sigma))} REV_k(s(\sigma), Q) = DL_k(s(\sigma), p(\sigma))$. ■

Theorem 10. For a SIMU-XOR auction, given auction state *k*, there can be no more than $min(N - x + 1, |\mathbb{P}|)$ live atomic bids with span *x*.

Proof. At auction state *k*, let the atomic bids of span *x* of each bidder be $\sigma_1, \sigma_2, \dots, \sigma_{|\mathbb{P}|}$. Without loss of generality, assume that $\sigma_{|\mathbb{P}|} \prec \sigma_{|\mathbb{P}|-1} \prec \cdots \prec \sigma_1$. Suppose $|\mathbb{P}| \geq N - x + 1$, consider the atomic bid σ_{N-x+1}, i.e., the $(N - x + 1)^{\text{th}}$ highest atomic bid at that span *x*. There are precisely $|\mathbb{P}| - (N - x) - 1$ atomic bids that are inferior to σ_{N-x+1}, namely $\sigma_{N-x+2}, \dots, \sigma_{|\mathbb{P}|}$. Therefore, for any bidder set Q such that $|Q| = |\mathbb{P}| - (N - x)$, it must contain *at least* one bidder who placed the atomic bid σ_t where $1 \leq t \leq N - x + 1$. Based on the assumed order, $\forall s$ such that $N - x + 2 \leq s \leq |\mathbb{P}|$, we have $\sigma_s \prec \sigma_t$. Because $\{\sigma_t\}$ itself is a feasible allocation for sub-auction $[x, Q]$, we know that either $\sigma_t = WIN_k[x, Q]$ or $\sigma_t \prec WIN_k[x, Q]$ is true. However, in either case, we always have $\sigma_s \prec WIN_k[x, Q]$. Because the choice of bidder set *Q* is arbitrary, it follows that $\sigma_s \in DEAD_k$. I.e., only atomic bids among $\sigma_1, \sigma_2, \dots, \sigma_{N-x+1}$ are possible to be live at auction state *k*. Suppose instead $|\mathbb{P}| < N - x + 1$, following the same logic above, we can see that all $|\mathbb{P}|$ atomic bids at span *x* can potentially be live. Overall, there cannot be more than $min(N - x + 1, |\mathbb{P}|)$ live atomic bids at span *x*. ■

Corollary 10. For a SIMU-XOR auction, given auction state *k*, the maximum possible number of live atomic bids is $\sum_{x=1}^{N} min(x, |\mathbb{P}|)$. I.e., $|LIVE_k| \leq \sum_{x=1}^{N} min(x, |\mathbb{P}|)$

Proof. Based on Theorem 10, by summing the live bids across all spans we have that $|LIVE_k| \leq \sum_{x=1}^{N} min(N - x + 1, |\mathbb{P}|) = \sum_{x=1}^{N} min(x, |\mathbb{P}|)$. ■

Appendix C

Additional Theoretical Results

Here we list two additional theoretical results for SIMU auctions, which are not discussed in the main paper but may provide further insights to understand SIMU auctions.

First, in SIMU-OR auctions, even by very quick, naïve calculations, one can show that can never have more than $O(N \cdot logN)$ live bids at a time, as explained below.

In SIMU-OR auctions, for any auction state k, $\lvert LIVE_k \rvert = O(N \cdot logN)$.

Proof. The proof of the lemma follows immediately from the fact that in an auction of size *N*, there can be no more than *N* live 1-item bids (i.e., bids with span 1), no more than *N*/2 live 2-item bids, or, more generally, no more than *N*/*x* live *x*-item bids, where $x = 1, 2, \dots, N$. And from elementary mathematical analysis we have that $N + \frac{N}{2} + \cdots + \frac{N}{N} = N \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) = O(N \cdot logN)$. ■

Note that the above result could be written as a non-asymptotic upper bound as well, i.e., without using *O*(·) notation. In such case, the naïve upper bound could be stated as: $\lvert LIVE_k \rvert \le \sum_{x=1}^{N} \lfloor N/x \rfloor$. The analogous result about the number of winning bids is obvious: $\lvert WIN_k \rvert \le N$, i.e., there can be at most *N* winning bids in an auction of size *N*, because each bid has to bid on at least one item.

Second, in a SIMU-XOR auction, if a bidder submits multiple atomic bids simultaneously, each of which is above the current winning level at its span, then only one of those atomic bids can be winning (due to the XOR constraint), and the winner will be the atomic bid with largest margin over its current winning level. If several atomic bids all have the largest margin, the one with smallest span will win (due to fairness-based tie-breaking). This is summarized as follows.

For a SIMU-XOR auction, given auction state k and a new general bid $b_{k+1} = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$, where $\forall x \in \{1,2,\dots,N\}$, $p(\sigma_x) = p^*$, $s(\sigma_x) = x$, and $v(\sigma_x) > WL_k(x, p^*)$. Let s^* be the ***smallest*** span s that satisfies $s \in \underset{x \in \{1,2,\dots,N\}}{\operatorname{argmax}} \left(v(\sigma_x) - WL_k(x, p^*)\right)$. Then $\sigma_{s^*} \in WIN_{k+1}$.

Proof. For atomic bid $\sigma_x \in b_{k+1}$, if it were to win the auction, the resulting winning allocation would be $\{\sigma_x\} \cup WIN_k(N - x, \mathbb{P} \backslash \{p^*\})$, which has the revenue of $v(\sigma_x) + REV_k(N - x, \mathbb{P} \backslash \{p^*\})$. Because $v(\sigma_x) + REV_k(N - x, \mathbb{P} \backslash \{p^*\}) = WL_k(x, p^*) + REV_k(N - x, \mathbb{P} \backslash \{p^*\}) + v(\sigma_x) - WL_k(x, p^*) = REV_k(N, \mathbb{P}) + v(\sigma_x) - WL_k(x, p^*)$, the maximum revenue is achieved by span s that satisfies $s \in \underset{x \in \{1,2,\dots,N\}}{\operatorname{argmax}} \left(v(\sigma_x) - WL_k(x, p^*)\right)$. The conclusion follows based on our tie-breaking mechanism. Furthermore, *only* σ_{s^*} in b_{k+1} will win in state $k + 1$ because of XOR bidding. ■

Appendix D

Benefits of a *Real-Time* Bidder Support System

In our paper, the difference between a real-time bidder support system (one that provides up-to-date information feedback after each bid) versus a non-real-time system (one that provides information feedback only at certain pre-specified intervals, i.e., which inevitably results in providing some outdated information) can have at least two implications for performance.

First, the two systems differ in their *usability/feasibility*. Because we consider continuous combinatorial auctions, where no formal "rounds of bidding" are imposed and the participations are completely asynchronous, it is critical to have a *real-time* bidder support system, simply because the auctions would be infeasible to conduct otherwise. Imagine an auction where bidders are free to join and leave as they wish, but have to wait (e.g., minutes or even hours) to obtain information feedback and construct their bids, such a mechanism will have staggeringly *low usability* and may not be adopted at all.

Second, the two systems can also result in different auction convergence outcomes and bidder experiences. While bidders can receive up-to-date information from a real-time system, they may receive obsolete information from a non-real-time system that provides feedback with some (potentially significant) delay, during which multiple bids could have been submitted. We illustrate this point using a numeric example and a set of stylized simulation examples. In our illustration, we assume bidders are relatively conservative and use deadness levels as bidding guidelines, and we show how a non-real-time system can result in slower auction convergence.

Numeric Example. Consider a SIMU-OR auction of 4 units with the following 4 OR bids already submitted:
$b_1 = (1, \$1)$, $b_2 = (1, \$1)$, $b_3 = (1, \$1)$, $b_4 = (2, \$4)$

At auction state 4 (i.e., after these 4 bids), the sub-auction revenues and deadness levels are as follows:
$REV_4(1) = \$1$, $REV_4(2) = \$4$, $REV_4(3) = \$5$, $REV_4(4) = \$6$
$DL_4(1) = \$1$, $DL_4(2) = \$2$, $DL_4(3) = \$5$, $DL_4(4) = \$6$

Now suppose a bidder wants to bid on 2 units and makes a request to see the deadness level for 2 units. Under a real-time bidder support system, the bidder sees the timely and correct deadness level of \$2 for 2 units, and can make informed bidding decisions accordingly. However, under a non-real-time bidder support system, it takes a longer time to update feedback metrics. For instance, suppose 2 bids are submitted during the time it takes to update feedback metrics, i.e., at auction state 4, the bidder still only has access to the deadness level calculated based on bids at auction state 2 (i.e., based on b_1 and b_2, deadness level for 2 units was \$0), which is already obsolete. Consequently, the bidder may place \$1 on 2 units, which appears to be live based on (obsolete) information calculated at state 2, but in fact is already dead based on (timely and correct) information at state 4. Therefore, having a non-real-time bidder support system that provides obsolete feedback information can lead bidders to make incorrect decisions (e.g., submitting bids that are already dead), which may slow down the convergence of the auction.

Simulation Experiments. Consider a SIMU-OR auction of 100 units and a set of bidders. We simulate both a real-time bidder support system and a non-real-time system as follows:

1. Real-time system: at auction state k, a bidder makes a bid by following two steps: (1) randomly pick span $s \in \{1, \ldots, 100\}$; (2) assign bid value $v = DL_k(s) + 1$.
2. Non-real-time system: assuming the system takes a longer time to update feedback metrics, during which t *new bids are submitted.* At auction state k, a bidder makes a bid by following two steps: (1) randomly pick span $s \in \{1, \ldots, 100\}$; (2) assign bid value $v = DL_{k-r-t}(s) + 1$, where $r = (k \bmod t)$ and $k - r - t$ represents the most recent state when deadness levels are updated. Also, for auction states $k < t$, bidders only have access to deadness levels at state 0.

In other words, under a real-time system, new bids have values higher than current deadness levels, and are live by construction. Under a non-real-time system, new bids have values higher than most recently updated deadness levels, which could be outdated. For simulation simplicity and illustrative purposes, we set a fixed auction revenue level of \$1000 and count the number of bids it takes to reach that level, as a measure of the speed of auction progression. We conduct 500 simulation runs and report two quantities: (1) the average number of bids submitted before auction revenue hits \$1000 and (2) the average percentage of bids that are already dead upon submission. In other words, the first quantity represents the speed of auction progression, and the second quantity reflects the amount of "wasted effort" in the auction. We report the results in the following table.

Update Speed	Average Number of Bids before Revenue Hits \$1000	Average Percentage of Dead Bids upon Submission
Real-time	3586.70	0%
$t = 1$	3976.21	13%
$t = 5$	6324.71	52%
$t = 10$	8559.97	68%
$t = 15$	10347.07	74%

Based on the results, as t grows larger (i.e., as the system becomes less real-time), we can see that it takes more bids to reach \$1000 auction revenue, and a higher percentage of those bids are already dead upon submission, because of the obsolete deadness level information.

In summary, compared to a real-time bidder support system, a non-real-time system may result in significantly slower auction progression and substantial "wasted effort", and hence slower auction convergence as well as potentially lower bidder satisfaction.

Appendix E

Comparison Between OR Versus XOR Bidding Language

Bidder's Perspective

From the bidder's perspective, the OR and XOR bidding languages differ on at least two aspects: *expressiveness* and *simplicity* (Nisan 2000).

<u>Expressiveness</u>: the XOR bidding language is strictly more expressive than the OR bidding language, i.e., any valuations expressed in OR bids can be expressed in XOR bids, and XOR language can also directly express substitutability among bids. For example, consider a SIMU auction of 10 units, we provide two canonical cases to illustrate this aspect.

Example E1: Suppose a bidder is willing to pay \$6 for 4 units or \$7 for 5 units, but has \$0 valuation for fewer than 4 units and \$7 valuation for more than 5 units. Under XOR language, the bidder can directly express the preference on 4 or 5 units by making 2 XOR bids, respectively \$6 on 4 units and \$7 on 5 units. Under OR language, the bidder *cannot* express this preference perfectly. By placing \$6 on 4 units and \$7 on 5 units, the bidder may end up having to pay \$13 for 9 units, despite her \$7 valuation for 9 units. Alternatively, by placing \$6 on 4 units and \$1 on 1 unit, the bidder may end up having to pay \$1 for 1 unit, despite her \$0 valuation for 1 unit. Either way, the bidder is exposed to the risk of disutility.

Example E2: Suppose a bidder has non-zero valuations for any number of units, and her valuations are increasing with diminishing margins with respect the number of units. I.e., let v_i represent her valuation for i units, then $\forall i$, $v_i > 0$ and $v_{i+1} - v_i < v_i - v_{i-1}$. Under XOR language, the bidder can directly express this preference by making 10 XOR bids, respectively v_i on i units. Under OR language, the bidder can express this preference, but must express it differently. In particular, she needs to make 10 OR bids, each on only 1 unit, with values $v_1, (v_2 - v_1), \ldots, (v_{10} - v_9)$. Suppose the bidder (mistakenly) places v_1 on 1 unit and v_2 on 2 units, she may end up having to pay $v_1 + v_2$ for 3 units, which is higher than her valuation for 3 units (v_3), because of the concavity of valuations.

Simplicity: the XOR language is less simple than OR language (i.e., it may take fewer OR bid elements to express certain preference than XOR). To see this, consider a simple SIMU auction of 3 units. If a bidder is willing to pay \$5 for 1 unit, \$10 for 2 units, and \$15 for 3 units, he/she only needs to make two OR bids (i.e., a \$5 bid on 1 unit and a \$10 bid on 2 units), but would have to make three XOR bids (i.e., a \$5 bid on 1 unit, a \$10 bid on 2 units, and a \$15 bid on 3 units).
Given these two differences, the bidding language used in an auction should be sufficiently expressive for its specific application needs and also simple for bidders to use.

Auctioneer's Perspective

Assuming the auctioneer's goal is to maximize auction revenue, which of the two bidding languages leads to higher revenue depends heavily on many different factors, including, at the very least: (1) bidders' valuations; (2) bidders' bidding behaviors (i.e., how they choose to express their preferences under different bidding languages); (3) auction progression (i.e., how bidders revise their bids during the course of an auction); and (4) other auction-specific activity rules. Below we construct 3 numeric examples to show, under some *specific assumptions about bidder valuations and bidding behaviors*, the OR language may result in higher or equal revenue than the XOR language. In these examples, we explicitly fix bidders' valuations, and assume they make bids to express their valuations in a one-shot manner (i.e., without iteratively revising their bids later). These stylized examples are intended to illustrate only a small portion of the potential intricacies of this problem.

Example E3: OR leads to higher revenue than XOR.
Consider a SIMU auction of 4 units and 2 bidders, A and B, with the following valuations:
- Bidder A is willing to pay \$5 for 2 units, \$8 for 3 units;
- Bidder B is willing to pay \$6 for 3 units or \$9 for 4 units.

Under OR bidding language, suppose bidders choose to make the following bids:
- Bidder A makes two OR bids: (\$3 on 1 unit) OR (\$5 on 2 units);
- Bidder B makes two OR bids: (\$3 on 1 unit) OR (\$6 on 3 units).

As the result, the auction revenue is \$11, by allocating 3 units to bidder A for price \$3+\$5 and allocating 1 unit to bidder B for price \$3.

On the other hand, under XOR language, suppose bidders choose to make the following bids:
- Bidder A makes two XOR bids: (\$5 on 2 units) XOR (\$8 on 3 units);
- Bidder B makes two XOR bids: (\$6 on 3 units) XOR (\$9 on 4 units).

As the result, the auction revenue is \$9, by allocating all 4 units to bidder B. In other words, the OR bidding language results in \$2 higher revenue than XOR language.

Example E4: OR leads to equal revenue as XOR
Consider the same auction in the previous example. Under OR bidding language, suppose bidders choose to make the following bids:
- Bidder A makes two OR bids: (\$5 on 2 units) OR (\$8 on 3 units);
- Bidder B makes two OR bids: (\$6 on 3 units) OR (\$9 on 4 units).

Since the combination of any two bids contains more than 4 units, which cannot be fulfilled, the auction revenue is \$9, by allocating all 4 units to bidder B. This is the same auction revenue under the XOR language.

Example E5: OR leads to equal revenue as XOR
Consider a SIMU auction of 3 units and 2 bidders, A and B. Each bidder is willing to pay \$3 for 1 unit, \$5 for 2 units, or \$6 for 3 units (i.e., both bidders have decreasing marginal valuations).

Under OR bidding language, suppose bidders choose to make the following bids:
- Bidder A makes three OR bids: (\$3 on 1 unit) OR (\$2 on 1 unit) OR (\$1 on 1 unit);
- Bidder B makes three OR bids: (\$3 on 1 unit) OR (\$2 on 1 unit) OR (\$1 on 1 unit).

As the result, the auction revenue is \$8, by allocating 2 units to bidder A for price \$3+\$2 (assuming bidder A bids first) and allocating 1 unit to bidder B for price \$3.

On the other hand, under XOR language, suppose bidders choose to make the following bids:
- Bidder A makes three XOR bids: (\$3 on 1 unit) XOR (\$5 on 2 units) XOR (\$6 on 3 units);
- Bidder B makes three XOR bids: (\$3 on 1 unit) XOR (\$5 on 2 units) XOR (\$6 on 3 units).

As the result, the auction revenue is also \$8, by allocating all 2 units to bidder A for price \$5 (assuming bidder A bids first) and allocating 1 unit to bidder B for price \$3.

Example E3 represents the scenario where bidders cannot express their valuations perfectly under OR language. Specifically, by making 1-unit bids, they are exposed to the risk of winning 1 unit, even if they may not want it. However, they can directly and perfectly express their valuations under XOR language. As such, the auction revenue under OR is higher than under XOR. Example E4 and E5 represent two special cases where bidders can perfectly express their valuations under both OR and XOR languages, and therefore result in the same auction revenue.

In addition to the above numerical examples, we also provide theoretical analyses of a stylized case, where OR language may lead to higher or equal auction revenue as XOR language.

Consider a SIMU auction of N units and 2 bidders, A and B, with the following valuations:
- Bidder A is willing to pay \$$a_x$ for x units or \$$a_y$ for y units;
- Bidder B is willing to pay \$$b_x$ for x units or \$$b_y$ for y units.

Furthermore, assume that $x < y < x + y \leq N < 2y$ and $2y - x \leq N$. Also assume that $a_x < a_y$, $b_x < b_y$, i.e., the bid value is higher for more units.

Under XOR language, bidders can perfectly express their preferences by making the following bids:
- Bidder A makes two XOR bids: (\$$a_x$ on x units) XOR (\$$a_y$ on y units);
- Bidder B makes two XOR bids: (\$$b_x$ for x units) XOR (\$$b_y$ for y units).

As a result, the auction revenue is $\max\{a_x + b_y, a_y + b_x\}$.

On the other hand, under OR bidding language, due to our assumptions on x and y, bidders cannot perfectly express their preferences without being exposed to the risk of acquiring allocations they don't value. Here, we consider four different situations:

<u>*Situation 1*</u>:
- Bidder A makes two OR bids: (\$$a_x$ on x units) OR (\$$a_y$ on y units).
- Bidder B makes two OR bids: (\$$b_x$ for x units) OR (\$$b_y$ for y units).

The resulting auction revenue is $\max\{a_x + a_y, a_x + b_y, a_y + b_x, b_x + b_y\}$.

<u>*Situation 2*</u>:
- Bidder A makes two OR bids: (\$$a_x$ on x units) OR (\$$(a_y - a_x)$ on $y - x$ units).
- Bidder B makes two OR bids: (\$$b_x$ for x units) OR (\$$(b_y - b_x)$ for $y - x$ units).

The resulting auction revenue is $\max\{a_x + b_x + (b_y - b_x), b_x + a_x + (a_y - a_x)\} = \max\{a_x + b_y, a_y + b_x\}$.

<u>*Situation 3*</u>:
- Bidder A makes two OR bids: (\$$a_x$ on x units) OR (\$$(a_y - a_x)$ on $y - x$ units).
- Bidder B makes two OR bids: (\$$b_x$ for x units) OR (\$$b_y$ for y units).

The resulting auction revenue is $\max\{a_x + b_y, b_x + a_x + (a_y - a_x), (a_y - a_x) + b_y\} = \max\{a_x + b_y, a_y + b_x, (a_y - a_x) + b_y\}$ (revenue of $(a_y - a_x) + b_y$ is possible because we have assumed $2y - x \leq N$).

<u>*Situation 4*</u>:
- Bidder A makes two OR bids: (\$$a_x$ on x units) OR (\$$a_y$ on y units).
- Bidder B makes two OR bids: (\$$b_x$ for x units) OR (\$$(b_y - b_x)$ for $y - x$ units).

The resulting auction revenue is $\max\{a_y + b_x, a_x + b_x + (b_y - b_x), (b_y - b_x) + a_y\} = \max\{a_y + b_x, a_x + b_y, (b_y - b_x) + a_y\}$ (revenue of $(b_y - b_x) + a_y$ is possible because we have assumed $2y - x \leq N$).

Overall, in situation 2, OR results in the same revenue as XOR; in situations 1, 3, and 4, OR results in no less revenue than XOR (i.e., OR may result in higher or equal revenue as XOR, depending on the specific values of a_x, a_y, b_x, b_y).

Generalizing from these cases, we argue that, assuming bidders make truthful bids based on their valuations in a one-shot manner, (1) OR language results in *no less* auction revenue than XOR *if bidders cannot perfectly express their preferences under OR*; and (2) OR language results in *equal* auction revenue than XOR *if bidders can perfectly express their preferences under OR*.

Importantly, in addition to bidding language choice, auction revenue depends on bidders' specific valuations and how they choose to express their valuations, including bidders' strategies for updating their bids continuously over time (i.e., in non-one-shot contexts). We would also like to emphasize that the choice of OR versus XOR has other implications in addition to revenue. For instance, in the above Example E3, bidder B is allocated 1 unit, which may be deemed undesirable by the bidder (the bidder only has non-zero valuations for 3 or 4 units). This can potentially lead to lower bidder satisfaction and discourage participation. Therefore, the auctioneer needs to consider multiple aspects (not just revenue) when choosing bidding language.