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HUMAN CAPITAL DEVELOPMENT FOR PROGRAMMERS USING OPEN SOURCE SOFTWARE

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Appendix

Proofs of Lemmas and Propositions

Proof of Lemma 1

The maximum value of w needed to satisfy the IR constraint of the programmer (called w_{IR}) is obtained from $w_{IR}T = M_0 - ax(T)|_{v=0}$. Here $x(T)|_{v=0}$ represents the smallest possible skill level with which the programmer ends the contract. This skill level occurs when $v = 0$ is chosen throughout the contract duration. Next, note that the adjoint equation for the co-state variable $\mu(t)$ is $\dot{\mu} = -\frac{\partial H}{\partial y} = 0 \Rightarrow \mu(t) = \text{constant} \ \forall\ t$.

Thus μ must take the same value throughout the contract duration. If $\mu > 1$, then $\frac{\partial H}{\partial w} > 0$ implying that w must be the maximum possible and hence the optimal value of $w > w_{IR}$, since the domain of the wage premium w is R^+. However, such a choice of w results in a smaller net value for the firm than if the firm chose $w = w_{IR}$ since paying wages is a cost to the firm and has no impact on programmer productivity. This is a contradiction because a suboptimal value of w gives a better solution to the firm. Hence it must be that $\mu \leq 1$. If $\mu < 1$, then $\frac{\partial H}{\partial w} < 0 \Rightarrow w = 0$. On the other hand, if $\mu = 1$, then $\frac{\partial H}{\partial w} = 0 \Rightarrow w$ can take any possible positive value (singular solution).

Proof of Lemma 2

We represent $\frac{\partial H}{\partial v}$ by z. Thus, using Equation 4, we can write

$$\lambda = \frac{z + Kx}{\alpha} - a\mu \tag{6}$$

Next, taking the derivative of both sides of Equation 4 with respect to t, we have

$$\dot{z} = -K\dot{x} + \dot{\lambda}\alpha + \lambda(\alpha_t + \alpha_x \dot{x}) + a\mu(\alpha_t + \alpha_x \dot{x})$$

Substituting in the value of $\dot{x}$ from Equation 1, and that of $\dot{\lambda}$ from the adjoint equation, $\dot{\lambda} = -\frac{\partial H}{\partial x} = -\left(K(1-v) + \lambda(v\alpha_x + \delta_x - \beta_x) + a\mu(v\alpha_x + \delta_x - \beta_x)\right)$, we can rewrite the above equation as

$$\lambda = \frac{K(\beta - \delta - \alpha) - \dot{z}}{\alpha_x(\beta - \delta) - \alpha_t - \alpha(\beta_x - \delta_x)} \tag{7}$$

Equating the values of λ from equations 6 and 7, we obtain

$$\dot{z} = z\left(-\frac{\alpha_x(\beta - \delta) - \alpha_t - \alpha(\beta_x - \delta_x)}{\alpha}\right) + \left(K(\beta - \delta - \alpha) - \frac{Kx\left(\alpha_x(\beta - \delta) - \alpha_t - \alpha(\beta_x - \delta_x)\right)}{\alpha}\right) \tag{8}$$

Note that the coefficient of z as well as the term independent of z are purely functions of x and and are independent of the controls or t. Hence, we represent these two terms by $-g(x,t)$ and $h(x,t)$, respectively.

Now suppose that $z = 0$ at two different instants τ_0 and τ_1, where $\tau_0 < \tau_1$, without loss of generality. We consider an instant t such that $\tau_0 < t < \tau_1$ and $z(t) \neq 0$. The general solution of the differential equation 8 is

$$z(t)\exp\int g(x,\sigma)d\sigma = \int \exp^{\int g(x,\sigma)d\sigma} h(x,s)ds + C$$

where C is the constant of integration. Using this equation we can write the solution between t and τ_1 as

$$z(\tau_1)\exp\int_0^{\tau 1} g(x,\sigma)d\sigma - z(t)\exp\int_0^{t} g(x,\sigma)d\sigma = \int_0^{\tau 1}(h(x,s))\exp\int_0^{S} g(x,\sigma)d\sigma\, ds$$

Since $z(\tau_1) = 0$, this can be rewritten as

$$z(t)\exp\int_0^{t} g(x,\sigma)d\sigma = -\int_0^{\tau 1}(h(x,s))\exp\int_0^{S} g(x\sigma)d\sigma\, ds \tag{9}$$

Similar to above, we consider the solution of the differential equation 8 between τ_0 and t. This gives us

$$z(t)\exp\int_0^{t} g(x,t) = -\int_t^{\tau_0}(h(x,s))\exp\int_0^{S} g(x\sigma)d\sigma\, ds \tag{10}$$

From equations 9 and 10, we note that $z(t)$ must have two different values at t. However, equation 8 implies that $z(t)$ must be a continuous function. Hence, it cannot have two values at any point. Hence our supposition that $z = 0$ at two non-contiguous instants must be incorrect. Since z is continuous in t, it can be either positive of negative prior to and after when it becomes zero. If $z > 0$, $v = 1$ and if $z < 0$, $v = 0$. This gives us the statement of the lemma.

Proof of Proposition 1

Statement A of the proposition follows from Lemma 2.

For Statement B, note that $\frac{\partial H}{\partial v} = 0$ and $\frac{d\frac{\partial H}{\partial v}}{dt} = 0$ are required for a singular solution since the partial derivative of the Hamiltonian with respect to the control must be zero and must continue to remain to be zero for some interval. Also, $\dot{\lambda} = -\frac{\partial H}{\partial x}$ is the standard adjoint condition. These three equations simplify to the following:

$$-Kx + \alpha(\lambda + a\mu) = 0 \tag{11}$$

$$-K\alpha + K(\beta - \delta) - (\lambda + a\mu)\big(\alpha_x(\beta - \delta) - \alpha_t - \alpha(\beta_x - \delta_x)\big) = 0 \tag{12}$$

$$\dot{\lambda} = -\big(K(1-v) + \lambda(v\alpha_x + \delta_x - \beta_x) + a\mu(v\alpha_x + \delta_x - \beta_x)\big) \tag{13}$$

Simultaneously solving equations 11 and 12 we obtain the solutions for x and λ, whereupon we can find $\dot{\lambda}$. Substituting in values of λ, $\dot{\lambda}$, and x in Equation 13, we obtain the value of v for the singular solution.

For Statement C, note that for $\mu = 0$, the IR constraint is not binding. Further, $\lambda(T) = 0$ since the firm does not have a salvage value of programmer skills. From Equation (4), we have $\frac{\partial H}{\partial v}|_{t=T} = -Kx < 0$. From Lemma 2, we know that $\frac{\partial H}{\partial v}$ does not change sign in $\hat{t}$ to T. Hence $v = 0$ in this interval. When $0 \le \mu \le 1$ (i.e., when IR binds), $\frac{\partial H}{\partial v}$ is either positive or negative in this interval. Correspondingly, $v = 1$ or $v = 0$.

Proof of Lemma 3

(A) In the proof of Lemma 2, from Equation 9, we see that the sign of the Hamiltonian, $z(t)$, is opposite to the sign of function $h(x,t)$ for $t < \bar{t}$. With the simplified state equation, we have $h(x,t) = \frac{-c_1c_2K - c_2^2K + 2c_2(c_4 - c_3)Kx + c_3(c_4 - c_3)Kx^2}{c_2 + c_3x}$. Using this we can easily show that $x(t) < \bar{x} \Rightarrow z(t) > 0 v = 1$ and $x(t) > \bar{x} \Rightarrow z(t) < 0 \Rightarrow v = 0$. By Lemma 2, it must be that $z(t)$ has the same sign in the interval 0 to $\bar{t}$. Accordingly, it is sufficient to check if $x_0 < \bar{x}$. If that is true then $v = 1$ and if not then $v = 0$.

(B) Applying the method outlined in proof of Proposition 1, we find the values of v, x, and λ in the singular region $(\bar{t} < t \le \hat{t})$. These values are indicated by $\bar{v}$, $\bar{x}$, and $\bar{\lambda}$, respectively.

Note that $\bar{v}$ and $\bar{x}$ are independent of μ. Consequently, the optimal values of v in the pre-singular and the singular region are unaffected by whether the IR constraint is binding or not.

(C) In the region $\hat{t} \le t \le T$, we know that $v = 0$ when the IR constraint is not binding (Proposition 1).

Suppose that $v = 1$ is a solution when the IR constraint is binding. The corresponding Hamiltonian is $H_1 = -w + \lambda(c_1 + c_2 + c_3x - c_4x) + \mu\big(w + a(c_1 + c_2 + c_3x - c_4x)\big)$. Using the adjoint equation $\dot{\lambda} = -\frac{\partial H_1}{\partial x}$. This implies that $\dot{\lambda} = (c_4 - c_3)(\lambda + a\mu)$. Note that the sign of $\dot{\lambda}$ crucially depends upon the value of λ at $t = \hat{t}$ (i.e., $\bar{\lambda}$). Also note that λ must be equal to 0 at $t = T$ since the firm has no salvage value for the skills of the programmer at $t = T$. If $\bar{\lambda} > 0$ or if $\bar{\lambda} < -a\mu 0$, then this is impossible (since then λ either monotonically increases from a positive value or monotonically decreases from a negative value). However this may be possible when $\bar{\lambda} < 0$ and $\bar{\lambda} + a\mu > 0$ (now λ can increase monotonically from a negative value and reach zero). It is easy to verify that the second condition always holds under requirements imposed on the parameters c_1, c_2, c_3, and c_4. The first condition requires $\mu > \frac{\left(c_1c_3 + c_2c_4 - \sqrt{c_2(c_4 - c_3)(c_1c_3 + c_2c_4)}\right)K}{ac_3(c_1c_3 + c_2c_4)}$. Note from the discussion preceding Lemma 1 that $\mu \le 1$. This requires $K \le \frac{ac_3(c_1c_3 + c_2c_4)}{c_1c_3 + c_2c_4 - \sqrt{c_2(c_4 - c_3)(c_1c_3 + c_2c_4)}}$. This is ruled out due to our restriction on marginal productivity of programmers. Hence, $v = 1$ cannot be a solution in this region when the IR constraint is binding.

Hence, $v = 0$, the only remaining possibility is the solution.

Proof of Proposition 2

The general solution to equation 5 is $x = \frac{c_1 + c_2v}{c_4 - c_2v} + \exp^{-(c_4 - c_3)v} C$, where C is the constant of integration. Using $x = x_0$ at $t = 0$ we obtain C. We know that $x = \bar{x}$ at $t = \bar{t}$. Using this relation, we get

$$\bar{t} = \frac{Log\left[\dfrac{(c_1c_3(c_3 - c_4) + c_2(c_3 - c_4)c_4 + \sqrt{c_2(c_4 - c_3)(c_1c_3 + c_2c_4)}(-c_4 + c_3v))(c_1 + c_2v - c_4x_0 + c_3vx_0)}{(c_1c_3 + c_2c_4)(c_1(c_3 - c_4) + c_2(c_4 - 2c_4v + c_3v^2))}\right]}{c_4 - c_3v}.$$

This expression is independent of μ. Hence it is applicable irrespective of whether the IR constraint is binding or not.

From Lemma 3, we know that $v(t) = 0 \,\forall\, t \in [\hat{t}, T]$. Now we write the differential equation for λ in this region. Using the adjoint equation, $\dot{\lambda} = -\frac{\partial H}{\partial x}$, this is

$$\dot{\lambda} = ac_4\mu + c_4\lambda - K \tag{14}$$

The general solution for this equation is $\lambda = -\frac{ac_4\mu - K}{c_4} + \exp^{c_4t} C$, where C is the constant of integration and t is transformed to the scale $[0, T - \hat{t}]$. Since, there is no salvage value of skills for the firm, we have $\lambda = 0$ at $t = T - \hat{t}$. Using this we obtain the expression for λ as a function of $\hat{t}$. This expression is used to find λ at $t = 0$ which is then equated with $\bar{\lambda}$ that we obtained in the proof for Lemma 3. This enables us to get $\hat{t} = T \dfrac{Log\left[\dfrac{(-c_1c_3(c_4 - c_3) - c_4(c_2(c_4 - c_3) + \sqrt{c_2(c_4 - c_3)(c_1c_3 + c_2c_4)}))(K - ac_4\mu)}{-(c_4 - c_3)(c_1c_3 + c_2c_4 - c_1c_4)K}\right]}{c_4}$. Note that $\bar{v} > 0$ requires $c_2 > \frac{c_1(c_4 - c_3)}{c_4}$. This condition makes the denominator of the *Log* term negative. Thus we can have a real value of $\hat{t}$ only when the numerator of the *Log* term is also negative. This will happen only when $K > ac_4\mu$. From discussion preceding Lemma 1 we know that $\mu \leq 1$. Note that $K > ac_4$ because of our restriction on marginal productivity of programmers and the requirement on parameters due to $\bar{v} > 0$. Hence $\hat{t}$ is real valued. Thus, since $K > ac_4\mu$, $\frac{\partial \hat{t}}{\partial \mu} = \frac{a}{K - ac_4\mu} > 0$, which is what we needed to establish.

Proof of Proposition 3

Consider a situation where the IR constraint binds but the firm does not pay any wage premium to the programmer ($w = 0$). In this situation, it must be that $ax(T) = M_0$. Using equation 5 and the boundary conditions $x = \bar{x}$ at $t = 0$ and $x = \frac{M_0}{a}$ at $t = T - \hat{t}$, and using the fact that $v = 0$ in between $\hat{t}$ and T, we find an expression for $\hat{t}_{IRB} = T + \dfrac{Log\left[\dfrac{(c_1c_3(c_3 - c_4) - c_4(c_2(c_4 - c_3) + \sqrt{c_2(c_4 - c_3)(c_1c_3 + c_2c_4)}))(ac_1 - c_4M_0)}{a(c_1c_3 + c_2c_4)(c_1c_3 + c_2c_4 - c_1c_4)}\right]}{c4}$.

Equating this expression with that $\hat{t}$ from proof of Proposition 2, we get an expression for μ in terms of M_0. It is algebraically tedious, but easy to show that μ is increasing in M_0. Using this property of μ, we can show that $0 \leq \mu \leq 1$ for $M_1 \leq M_0 \leq M_2$. Note that when $\mu = 1$, $\frac{\partial H}{\partial w} < 0$ and so $w = 0$. However, if $M_0 > M_2$ then $\mu = 1 \Rightarrow \frac{\partial H}{\partial w} = 0$, and hence the firm can pay a positive wage w.

Proof of Proposition 4

Part 1

Suppose $x_0 > \bar{x}$. From Lemma 3, we know that $\frac{\partial H}{\partial v}|_{t=T} < 0$. Now either $\frac{\partial H}{\partial v} < 0 \,\forall\, t \in [0, T]$ implying $v(t) = 0$, or else there must be some τ such that $0 < \tau < T$ where $\frac{\partial H}{\partial v}|_{t=\tau} = 0$. Suppose that such a τ exists. Note that since the singular solution is ruled out, $\frac{\partial H}{\partial v}$ can be 0 at most at some instant but not in an interval. Then as per Lemma 2, $v(t) = 0 \ \forall\, t \in [\tau, T]$. Further, as per Lemma 3, $v(t) = 0 \ \forall\, t \in [0, \tau]$. Hence, $v(t) = 0$ throughout the contract duration is the only solution. This also shows that the firm never uses the option of training the programmers even if the IR constraint binds.

Part 2

Suppose $x_0 < \bar{x}$ and that there exists some τ such that $0 < \tau < T$ where $\frac{\partial H}{\partial v}\big|_{t=\tau} = 0$. As in Part 1, we know that singular solution is not possible, hence $\frac{\partial H}{\partial v}$ can be 0 at most at some instant but not in an interval. Then using Lemma 3, we know that $v(t) = 1$ for $t \in [0,\tau]$ and $v(t) = 0$ for $t \in [\tau,T]$. To find τ, we first consider the region $\tau < t < T$. Using the adjoint equation, $\dot{\lambda} = -\frac{\partial H}{\partial x}$ and the end condition $\lambda = 0$ at $= T$, we work out $\lambda(t) = \frac{K - ac_4\mu}{c_4} - \frac{\exp^{-c_4(T-t+\tau)}(K - ac_4\mu)}{c_4}$. We represent x at $t = \tau$ by x_τ. Using this as the boundary condition, and using the state equation 5, we find $x(t) = \frac{\exp^{c_4 t}(c_4 x_\tau - c_1) + c_1}{c_4}$. Using the expressions for $x(t)$ and $\lambda(t)$, we can work out the expression for $\frac{\partial H}{\partial v}$. Now utilizing $\frac{\partial H}{\partial v}\big|_{t=\tau} = 0$, we find $x_\tau = \frac{c_2 \exp^{c_4\tau}\left((-1 + \exp^{c_4(T-\tau)})K + ac_4\mu\right)}{(c_4 - c_3)\exp^{c_4 T} K + c_3 \exp^{c_4\tau}(K = ac_4\mu)}$.

Now we consider the region $0 < t < \tau$. Using the state equation 5 and the boundary condition $x(0) = x_0$, we can work out the expression for $x(t)$. Now, τ can be obtained from solving

$$x(t)|_{t=\tau} = x_\tau \tag{15}$$

This is the equation in the statement of the proposition.

To show that τ is unique, we take the derivative of x_τ w.r.t τ. This is easily seen to be negative. Next we take the derivative of $x(t)|_{t=\tau}$ w.r.t τ and this expression shows that its minimum value occurs at $x_0 = \bar{x}$. This minimum value of the slope is found to be positive. Thus the expressions on the left hand and right hand side of the equation above have opposite signs w.r.t. τ. Thus τ must be unique.

Finally, if $\tau < 0$, then $v = 0$ for $0 \le t \le T$.

To see how τ behaves as the IR constraint binds, we rewrite equation 15 as $x(t)|_{t=\tau} - x_\tau = 0$ and represent the LHS this equation by f. Then taking the derivative w.r.t. μ we have $\frac{\partial f}{\partial \mu} + \frac{\partial f}{\partial \tau}\frac{d\tau}{d\mu} = 0$. It is easy to show that $\frac{\partial f}{\partial \mu} < 0$ and $\frac{\partial f}{\partial \tau} > 0$. Thus, it must be that $\frac{d\tau}{d\mu} > 0$. Clearly as μ increases, τ also increases leading to a greater training period at the beginning of the contract duration. This leads to higher skills at the conclusion of training, thus enabling the firm to pay lower wages. However, we know that $\mu \le 1$. Hence, once μ reaches 1, no further extension in τ is possible and any shortfall in IR constraint must then be paid through a wage premium.